Presentations for Abstract Context Institutions *

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Abstract. The paper discusses a generalization of the notion of *context institution* and introduces a suitable notion of *presentation* for it. With an appropriate notion of morphism presentations constitute a category, whose structural properties can be used to systematically construct logical systems (i.e. abstract context institutions).

1 Introduction

Context institutions, as introduced in [12], enrich the inner structure of institutions (cf. [6, 19]) by adding notions such as *contexts* and *substitutions*.

Context institutions are "concrete" in a sense similar to *concrete institutions* of Bidoit and Tarlecki [1]—for every signature, the category of models is concrete over the category of *indexed sets*, and consequently, there is a notion of a *carrier* for each model. Similarly contexts "contain" *sorted sets* of variables. The *satisfaction relation* relates *(open) formulae* over a given context and *valuations*. The latter are just functions from variables to carriers.

In the present paper we shall generalize context institutions by dropping the "concreteness" assumption. The resulting notion of *abstract context institution* has a simpler definition, while retaining the structural power of the original notion, and allowing for a wider range of examples.

We shall also introduce a notion of *presentation* for abstract context institutions. Presentations, also known as *parchments*, have been originally invented as a tool for proving the *satisfaction condition* for institutions (cf. [5]). In [18], parchments have been proposed as a framework for systematic construction of logical systems. This idea has been further developed in a series of papers [9–11].

Structurally, presentations for abstract context institutions are similar to the *model-theoretic parchments* of [11]. Internally however, they are based on a suitable notion of *metastructure* (see also [13]), rather than many-sorted algebras. The added "flexibility" of metastructures allows one to define certain useful constructions, which are not possible at the level of (model-theoretic) parchments.

After presenting some preliminaries, in Section 3 we shall define the notion of abstract context institution, giving also an informal motivation for its various

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components. We shall illustrate the introduced notion by means of an example. Then, specializing the notion of *institution morphism* (cf. [6]), we shall define the category of abstract context institutions and briefly discuss its properties.

In Section 4 we shall introduce *presentations* for abstract context institutions and give several examples as an illustration. In Section 5 we shall define a notion of *presentation morphism*, describe properties of categories of presentations and show how they can be used for structural construction of logics (abstract context institutions). Finally, in Section 6 we shall present some concluding remarks.

2 Preliminaries

In what follows we assume that the Reader is familiar with the basic notions of category theory (cf. [7, 2]) and universal algebra (cf. [3, 8]). Therefore the purpose of this section is mainly to fix notation being used throughout the rest of the paper and to introduce some less standard constructions.

Indexed and Sorted Sets. Let S be a set. The category of *S-indexed sets* $\mathbf{ISet}[S]$ is a category having S-indexed families of sets $\langle X_s \rangle_{s\in S}$ as objects, and S-indexed functions, i.e., families $\langle h_s : X_s \to Y_s \rangle_{s\in S}$ as morphisms. In fact, indexed sets form an *indexed category* $\mathbf{ISet} : \mathbf{Set}^{op} \to \mathbf{Cat}$ (cf. [20]).

For every $S \in |\mathbf{Set}|$, the full subcategory of $\mathbf{ISet}[S]$ determined by the following condition:

$$\forall X \in |\mathbf{SSet}[S]| \quad \forall s_1, s_2 \in S \quad s_1 \neq s_2 \;\Rightarrow\; X_{s_1} \cap X_{s_2} = \emptyset$$

will be called the category of *S-sorted sets*, and denoted by $\mathbf{SSet}[S]$.

The correspondence $S \mapsto \mathbf{SSet}[S]$, in an obvious way, extends to a functor $\mathbf{SSet} : \mathbf{Set} \to \mathbf{Cat}$.

For every function $f : S \to S'$, every S-sorted set X and every S'-indexed set V, there is a bijection:

$$(\!|\, \_ \,|\!)_f^{X,V} : \mathbf{ISet}[S](X, \mathbf{ISet}_f(V)) \to \mathbf{ISet}[S'](\mathbf{SSet}_f(X), V).$$

This bijection is actually natural in both X and V.[1]

Composition in Categories. In most cases the composition of morphisms f and g in a category will be denoted by $f\,;g$ (we shall always use the diagrammatic order). In the case of natural transformations, their *vertical* composition will be denoted by $\alpha\,;\beta$, and the *horizontal* one by $\gamma * \delta$.

[1] In other words $(\!|\, \_ \,|\!)_f^{\_,\_} : \mathbf{ISet}[S](\_, \mathbf{ISet}_f(\_)) \Rightarrow \mathbf{ISet}[S'](\mathbf{SSet}_f(\_), \_)$ is a natural isomorphism between the appropriately defined "generalized hom-functors".

Variables and Categories of Substitutions. In what follows, the category of (many-sorted) algebraic signatures will be denoted by **AlgSig**. Let *Var* be an infinite (countable) *vocabulary of variable symbols*. We shall assume, that *Var* comes with a fixed *choice function* i.e., a function $choice : (\mathcal{P}(Var) \setminus \{\emptyset\}) \to Var$, s.t. $choice(V) \in V$, for every non-empty set of variables V.

For every algebraic signature $\Sigma = \langle S, \Omega \rangle$, by the *category of Σ-substitutions*, we shall mean a category $\mathcal{T}_{\mathbf{\Sigma}}$ defined as follows:

- objects: finite S-sorted sets of elements of *Var*, where "X is finite" means, that $\bigcup\{ X_s \mid s \in S \}$ is finite;
- morphisms: a morphism $f : X \to Y$ is an arbitrary S-sorted function $f : X \to |T_\Sigma(Y)|$, where $T_\Sigma(Y)$ denotes the algebra of Σ-terms with variables from Y.

For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{T}_{\mathbf{\Sigma}}$, their composition is defined by $f;g \mathrel{\widehat{=}} f;|g^\sharp|$, where $g^\sharp$ is the free extension of g to a Σ-homomorphism.

The construction of the category $\mathcal{T}_{\mathbf{\Sigma}}$ can be extended to a functor $\mathcal{T}_{\_} : \mathbf{AlgSig} \to \mathbf{SCat}$, where **SCat** denotes the category of *small categories*. The construction of $\mathcal{T}_{\_}$ is pretty standard. Perhaps the only nontrivial part consists of showing that $\mathcal{T}_{\_}$ preserves composition (cf. [14]).

Categories of Diagrams. Let $\mathbf{K}$ be an arbitrary category. The category of *small diagrams in* $\mathbf{K}$ is a category $\mathbf{sDgm}(\mathbf{K})$, whose objects are pairs $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a *small* category and $F : \mathbf{A} \to \mathbf{K}$ is a functor. A morphism from $\langle \mathbf{A}, F \rangle$ to $\langle \mathbf{B}, G \rangle$ is a pair $\langle H, \alpha \rangle$, such that $H : \mathbf{A} \to \mathbf{B}$ is a functor and $\alpha : F \Rightarrow H\,; G$ is a natural transformation. Composition of morphisms $\langle H_1, \alpha \rangle : \langle \mathbf{A}_1, F_1 \rangle \to \langle \mathbf{A}'_1, F'_1 \rangle$ and $\langle H_2, \beta \rangle : \langle \mathbf{A}_2, F_2 \rangle \to \langle \mathbf{A}'_2, F'_2 \rangle$ is defined as $\langle H_1; H_2,\ \alpha\,; (H_1 * \beta) \rangle$.

For every category $\mathbf{K}$, there is a "projection" $\mathbf{Pr_K} : \mathbf{sDgm}(\mathbf{K}) \to \mathbf{SCat}$. This projection is a functor defined by: $\mathbf{Pr_K}(\langle \mathbf{A}, F \rangle) \mathrel{\widehat{=}} \mathbf{A}$ and $\mathbf{Pr_K}(\langle H, \alpha \rangle) \mathrel{\widehat{=}} H$. Using it, for any category $\mathbf{A}$ and any functor $G : \mathbf{A} \to \mathbf{sDgm}(\mathbf{K})$, we obtain a functor $\mathbf{bas}(G) : \mathbf{A} \to \mathbf{SCat}$ given by the composition $G\,; \mathbf{Pr_K}$. Similarly, for any natural transformation $\alpha : G_1 \Rightarrow G_2$ we obtain a natural transformation $\mathbf{bas}(\alpha) : G_1\,; \mathbf{Pr_K} \Rightarrow G_2\,; \mathbf{Pr_K}$ given by the composition $\alpha * \mathbf{Pr_K}$.

We shall call $\mathbf{bas}(G)$ and $\mathbf{bas}(\alpha)$ the *base for* F and the *base for* α respectively. The *base construction* actually defines a functor between appropriate functor categories.

Elements Construction. Let $\mathbf{CoFun}(\mathbf{Class})$ denotes the category whose objects are *contravariant functors* $F : \mathbf{C}^{op} \to \mathbf{Class}$, where $\mathbf{C}$ is an arbitrary (not necessarily small) category, and **Class** is the category of classes (i.e. "potentially large" sets). A morphism from $F : \mathbf{C}^{op} \to \mathbf{Class}$ to $G : \mathbf{D}^{op} \to \mathbf{Class}$ is a pair $\langle H, \alpha \rangle$, such that $H : \mathbf{C} \to \mathbf{D}$ is a functor and $\alpha : F \Rightarrow H^{op}\,; G$ is a natural transformation.

Then, there is a functor $\mathbf{Elts} : \mathbf{CoFun}(\mathbf{Class}) \to \mathbf{Cat}$ given by:

– *action on objects*: $\mathbf{Elts}(F : \mathbf{C}^{op} \to \mathbf{Class})$ is a category whose objects are pairs of the form $\langle C, c \rangle$, where $C \in |\mathbf{C}|$ and $c \in F(C)$. $f : \langle C_1, c_1 \rangle \to \langle C_2, c_2 \rangle$ is a morphism in $\mathbf{Elts}(F)$ if $f : C_1 \to C_2$ in $\mathbf{C}$ and $F(f)(c_2) = c_1$,
– *action on morphisms*: for any morphism $\langle H, \alpha \rangle : F \to G$ in the category $\mathbf{CoFun}(\mathbf{Class})$, the functor $\mathbf{Elts}(\langle H, \alpha \rangle)$ is given by: $\langle C, c \rangle \mapsto \langle H(C), \alpha_C(c) \rangle$ and $f \mapsto H(f)$.

For any functor $F : \mathbf{C}^{op} \to \mathbf{Class}$, the category $\mathbf{Elts}(F)$ is usually called the *category of elements for* F (cf. [2]). By $\mathbf{ind}(F)$ we shall denote the obvious "projection" functor from $\mathbf{Elts}(F)$ to $\mathbf{C}$.

3 Abstract Context Institutions

3.1 Definition

Signatures. Similarly as in the case of ordinary institutions, abstract context institutions are built around the notion of *signature*. Signatures provide a *vocabulary* for constructing formulae. Since in computer science applications it is often desirable to be able to change the notation being used, signatures are required to form a category. This category will usually be denoted by **Sig** (possibly with a suitable superscript).

Contexts, Substitutions and Formulae. For every signature Σ, we want to consider (possibly *open*) *formulae* over Σ. Therefore we shall assume that for every Σ, there is a (small) category of Σ*-contexts* $\mathbf{Ctxt}_\Sigma$. Context morphisms are meant to model *substitutions*.

The fact that for every Σ-context we have a corresponding set of formulae (built "over" that context), will be modeled by a functor $\mathbf{Frm}_\Sigma : \mathbf{Ctxt}_\Sigma \to \mathbf{Set}$. For any context morphism (substitution), its image under $\mathbf{Frm}_\Sigma$, is a function "performing" the substitution.

To take the change of notation into account, we shall eventually define the *formula functor* as a functor $\mathbf{Frm} : \mathbf{Sig} \to \mathbf{sDgm}(\mathbf{Set})$.

Models and Valuations. The model structure of abstract context institutions will be given, as in the case of institutions, by a functor $\mathbf{Mod} : \mathbf{Sig}^{op} \to \mathbf{Class}$. The "semantical part" of abstract context institutions will also contain the notion of *valuations*.

Let us take an arbitrary signature Σ and model $M \in \mathbf{Mod}_\Sigma$. We shall assume, that for every Σ-context Γ there is a set $\mathbf{Val}_{\Sigma,M}(\Gamma)$, whose elements will be called *valuations of* Γ *in* M. In typical cases valuations are just (suitably indexed) functions from (similarly sorted) sets of variables into the "carrier" of the model.

To better motivate the remaining parts of the valuation structure let us look at very simple example. Let us assume that our contexts are just sets of variables, models are sets of natural numbers and valuations are total functions between them. Let us take two contexts $\{x\}$ and $\{y\}$ and a context morphism (i.e., substitution) t defined by $t(x) = 2 * y$. For an arbitrary valuation of the context $\{y\}$—e.g., $[y \mapsto 5]$, we can "evaluate" the substitution t to get the "corresponding" valuation of the context $\{x\}$, which in this case will be $[x \mapsto 10]$.

In a general case it means, that for every context morphism $t : \Gamma \to \Delta$ there is a function $\mathbf{Val}_{\Sigma,M}(t) : \mathbf{Val}_{\Sigma,M}(\Delta) \to \mathbf{Val}_{\Sigma,M}(\Gamma)$ between the corresponding sets of valuations. It seems natural to require that "evaluation" of a composition of two substitutions has the same effect as a composition of their evaluations. We shall ensure this by requiring the functoriality of $\mathbf{Val}_{\Sigma,M} : \mathbf{Ctxt}^{op} \to \mathbf{Set}$.

As in the case of formulae, every signature morphism "generates" a corresponding "translation" of valuations. To illustrate this, let us assume that our category of signatures is the category of sets **Set**. As contexts lets us take sorted sets of variables (elements of a fixed vocabulary of variable symbols), and as models—indexed sets of natural numbers.

Let $S = \{s_1, s_2\}$ and $S' = \{s'\}$ be signatures. Let us take a signature morphism $\sigma : S \to S'$ given by $\sigma(s_1) = \sigma(s_2) = s'$, and an arbitrary S'-model (i.e., an S'-indexed set of natural numbers)—for example $M' = \{3, 7\}$. The *reduct* of M' under $\mathbf{Mod}_\sigma$ is an S-indexed set M defined by: $M_{s_1} = M_{s_2} = \{3, 7\}$.

Let X given by $X_{s_1} = \{x\}$, $X_{s_2} = \{y\}$ be a S-context. Let us consider a valuation $v \in \mathbf{Val}_{S,M}(X)$ s.t. $v = \langle\, [x \mapsto 3]_{s_1}, [y \mapsto 7]_{s_2} \,\rangle$. In an obvious way we can translate v "along" σ, obtaining a valuation $v' \in \mathbf{Val}_{S',M'}(\{x, y\})$ $(\{x, y\} = \mathbf{Ctxt}_\sigma(X))$ s.t. $v' = \langle\, [x \mapsto 3, y \mapsto 7]_{s'} \,\rangle$. The valuation v' was obtained from v by suitable "reindexing" corresponding to the signature morphism σ.

Similarly, in a general case, we shall assume that for every: signature morphism $\sigma : \Sigma \to \Sigma'$, Σ-context Γ and Σ'-model M' there is a function, denoted by $\mathbf{val}_\sigma(\Gamma)$ which for every Σ-valuation from the set $\mathbf{Val}_{\Sigma,\mathbf{Mod}_\sigma(M')}(\Gamma)$ assigns a Σ'-valuation from $\mathbf{Val}_{\Sigma',M'}(\mathbf{Ctxt}_\sigma(\Gamma))$. Using the notion of the *category of elements* (see Sect. 2) we can formalize it requiring that the *valuation structure* of an abstract context institution is given by a functor $\mathbf{Val} : \mathbf{Elts}(\mathbf{Mod}) \to \mathbf{sDgm}(\mathbf{Set})$, such that for every object $\langle\, \Sigma, M \,\rangle$ in $\mathbf{Elts}(\mathbf{Mod})$ the domain of $\mathbf{Val}_{\Sigma,M}$ is the category $\mathbf{Ctxt}^{op}_\Sigma$ (see the *coherence condition* below).

Satisfaction Relation. For every signature Σ, model $M \in \mathbf{Mod}_\Sigma$ and Σ-context Γ, the *satisfaction relation* $M[\,\_\,] \models_{\Sigma,\Gamma} \_$ will be a binary relation between the set of valuations $\mathbf{Val}_{\Sigma,M}(\Gamma)$ and the set of formulae $\mathbf{Frm}_\Sigma(\Gamma)$.

The structural components of an abstract context institution are tight together by two "semantical" conditions: the *satisfaction condition* and the *substitution condition.* The former expresses that fact that the satisfiability of a formula is invariant under the change of notation (i.e., signature morphisms), and the latter—that it is also invariant under substitutions (i.e., context morphisms).

Definition 1. *A* context institution $\mathfrak{C}$ *consists of:*

- *a category* $\mathbf{Sig}^{\mathfrak{C}}$ *of* signatures*,*
- *a* formula functor $\mathbf{Frm}^{\mathfrak{C}} : \mathbf{Sig}^{\mathfrak{C}} \to \mathbf{sDgm}(\mathbf{Set})$*, whose base (see Sect. 2) will be called the* context functor *for* $\mathfrak{C}$ *and denoted by* $\mathbf{Ctxt}^{\mathfrak{C}}$ *(hence* $\mathbf{Ctxt}^{\mathfrak{C}} : \mathbf{Sig}^{\mathfrak{C}} \to \mathbf{SCat}$*),*

– *a* model functor $\mathbf{Mod}^{\mathfrak{C}} : (\mathbf{Sig}^{\mathfrak{C}})^{op} \rightarrow \mathbf{Class}$*,*
– *a* valuation functor $\mathbf{Val}^{\mathfrak{C}} : \mathbf{Elts}(\mathbf{Mod}^{\mathfrak{C}}) \rightarrow \mathbf{sDgm}(\mathbf{Set})$ *and*
– *for every: signature* Σ*, model* $M \in \mathbf{Mod}^{\mathfrak{C}}_{\Sigma}$*, context* $\Gamma \in |\mathbf{Ctxt}^{\mathfrak{C}}_{\Sigma}|$*, a binary* satisfaction relation*:*

$$M[\_] \models^{\mathfrak{C}}_{\Sigma,\Gamma} \_ \ \subseteq \ \mathbf{Val}^{\mathfrak{C}}_{\Sigma,M}(\Gamma) \times \mathbf{Frm}^{\mathfrak{C}}_{\Sigma}(\Gamma)$$

such that the following three conditions are satisfied:

– Coherence Condition: $\mathbf{bas}(\mathbf{ind}(\mathbf{Mod}^{\mathfrak{C}})\,;\mathbf{Frm}^{\mathfrak{C}}) = \mathbf{bas}(\mathbf{Val}^{\mathfrak{C}})\,;(\_)^{op}$ *this condition says that for every signature* Σ *and model* $M \in \mathbf{Mod}^{\mathfrak{C}}_{\Sigma}$*,* $\mathbf{Val}^{\mathfrak{C}}_{\Sigma,M} : (\mathbf{Ctxt}^{\mathfrak{C}}(\Sigma))^{op} \rightarrow \mathbf{Set}$*, and that for every signature morphism* $\sigma : \Sigma \rightarrow \Sigma'$*, the first element of the pair* $\mathbf{Val}^{\mathfrak{C}}_{\sigma}$ *is a functor* $(\mathbf{Ctxt}^{\mathfrak{C}}_{\sigma})^{op} : (\mathbf{Ctxt}^{\mathfrak{C}}_{\Sigma})^{op} \rightarrow (\mathbf{Ctxt}^{\mathfrak{C}}_{\Sigma'})^{op}$*,*
– Satisfaction Condition: *for every: morphism* $\sigma : \langle\, \Sigma, M \,\rangle \rightarrow \langle\, \Sigma', M' \,\rangle$ *in* $\mathbf{Elts}(\mathbf{Mod}^{\mathfrak{C}})$ *(i.e.,* $\sigma : \Sigma \rightarrow \Sigma'$ *in* $\mathbf{Sig}^{\mathfrak{C}}$ *s.t.* $M = \mathbf{Mod}^{\mathfrak{C}}_{\sigma}(M')$*), context* $\Gamma \in |\mathbf{Ctxt}^{\mathfrak{C}}_{\Sigma}|$*, valuation* $v \in \mathbf{Val}^{\mathfrak{C}}_{\Sigma,M}(\Gamma)$ *and formula* $\phi \in \mathbf{Frm}^{\mathfrak{C}}_{\Sigma}(\Gamma)$*:*

$$M[v] \models^{\mathfrak{C}}_{\Sigma,\Gamma} \phi \quad \textit{iff} \quad M'[\mathbf{val}^{\mathfrak{C}}_{\sigma}(\Gamma)(v)] \models^{\mathfrak{C}}_{\Sigma',\mathbf{Ctxt}^{\mathfrak{C}}_{\sigma}(\Gamma)} \mathbf{frm}^{\mathfrak{C}}_{\sigma}(\Gamma)(\phi),$$

where the natural transformations $\mathbf{val}^{\mathfrak{C}}_{\sigma} : \mathbf{Val}^{\mathfrak{C}}_{\Sigma,M} \Rightarrow (\mathbf{Ctxt}^{\mathfrak{C}}_{\sigma})^{op}\,;\mathbf{Val}^{\mathfrak{C}}_{\Sigma',M'}$ *and* $\mathbf{frm}^{\mathfrak{C}}_{\sigma} : \mathbf{Frm}^{\mathfrak{C}}_{\Sigma} \Rightarrow \mathbf{Ctxt}^{\mathfrak{C}}_{\sigma}\,;\mathbf{Frm}^{\mathfrak{C}}_{\Sigma'}$ *denote the second element of* $\mathbf{Val}^{\mathfrak{C}}_{\sigma}$ *and the second element of* $\mathbf{Frm}^{\mathfrak{C}}_{\sigma}$ *respectively.*
– Substitution Condition: *for every:* $\langle\, \Sigma, M \,\rangle \in |\mathbf{Elts}(\mathbf{Mod}^{\mathfrak{C}})|$*, context morphism* $f : \Gamma \rightarrow \Delta$*, formula* $\phi \in \mathbf{Frm}^{\mathfrak{C}}_{\Sigma}(\Gamma)$ *and valuation* $v \in \mathbf{Val}^{\mathfrak{C}}_{\Sigma,M}(\Delta)$*:*

$$M[\mathbf{Val}^{\mathfrak{C}}_{\Sigma,M}(f)(v)] \models^{\mathfrak{C}}_{\Sigma,\Gamma} \phi \quad \textit{iff} \quad M[v] \models^{\mathfrak{C}}_{\Sigma,\Delta} \mathbf{Frm}^{\mathfrak{C}}_{\Sigma}(f)(\phi).$$

Due to the lack of space, we shall illustrate the introduced notion by giving only one simple example—the abstract context institution of (many-sorted) *equational logic*. Many other examples, among them of partial, modal, and higher-order logics, can be found in [14]. Also examples of context institutions from [12] can be viewed as abstract context institutions.

Informally speaking, one can claim that every logical system with a "reasonable" notion of *substitution*, satisfying a *substitution lemma* can be represented by an abstract context institution. By *substitution lemma* we mean a property saying that for every substitution f, model M and formula ϕ of the given logic:

$$M \models \phi \quad \text{iff} \quad M \models f(\phi)$$

where $f(\phi)$ denotes a formula obtained from ϕ by performing the substitution f. The property expressef by the substitution lemma is closely related to the *substitution condition.*

3.2 Example: Many-Sorted Equational Logic

The equational logic example will not be presented in full detail, but we hope that the Reader will be able to easily fill the gaps wherever necessary.

The category of signatures $\mathbf{Sig}^{\mathrm{EL}}$ of the abstract context institution EL for the many-sorted equational logic is the category **AlgSig** of (many-sorted) algebraic signatures and their morphisms. The *context functor* $\mathbf{Ctxt}^{\mathrm{EL}}$ is the functor $\mathcal{T}_{\_} : \mathbf{AlgSig} \to \mathbf{SCat}$, defined in Sect. 2.

For a given Σ-context X the set of formulae $\mathbf{Frm}^{\mathrm{EL}}_{\Sigma}(X)$ is the set of Σ-equations with variables from X. Both the Σ-formula functor $\mathbf{Frm}^{\mathrm{EL}}_{\Sigma} : \mathbf{Ctxt}^{\mathrm{EL}}_{\Sigma} \to \mathbf{Set}$ and the formula functor $\mathbf{Frm}^{\mathrm{EL}} : \mathbf{AlgSig} \to \mathbf{sDgm}$ are defined in the usual way (cf. [12]). In particular, for any algebraic signature morphism $\sigma : \Sigma \to \Sigma'$, $\mathbf{Frm}^{\mathrm{EL}}_{\sigma}$ is a pair $\langle \mathbf{Ctxt}^{\mathrm{EL}}_{\sigma}, \mathbf{frm}^{\mathrm{EL}}_{\sigma} \rangle$, where $\mathbf{frm}^{\mathrm{EL}}_{\sigma} : \mathbf{Frm}^{\mathrm{EL}}_{\Sigma} \Rightarrow \mathbf{Ctxt}^{\mathrm{EL}}_{\sigma} ; \mathbf{Frm}^{\mathrm{EL}}_{\Sigma'}$ is a natural transformation given by:

$$\mathbf{frm}^{\mathrm{EL}}_{\sigma}(X)(t_1 \equiv t_2) \mathrel{\widehat{=}} |(\iota^{\sigma}_X)^{\sharp}|(t_1) \equiv |(\iota^{\sigma}_X)^{\sharp}|(t_2),$$

where $\iota^{\sigma}_X : X \to \mathbf{ISet}_{\sigma}(|T_{\Sigma'}(\mathbf{SSet}_{\sigma}(X))|)$[2] is the obvious inclusion.

The model functor $\mathbf{Mod}^{\mathrm{EL}} : \mathbf{AlgSig} \to \mathbf{Class}$ assigns to every algebraic signature Σ the class of all Σ-algebras, and to every algebraic signature morphism $\sigma : \Sigma \to \Sigma'$—the usual σ-reduct operation.

For any Σ-context X and every Σ-algebra A the set $\mathbf{Val}^{\mathrm{EL}}_{\Sigma,A}(X)$ is the set of all (S-indexed) functions from X to $|A|$—the *carrier* of A. For any Σ-context morphism (substitution) $f : X \to Y$ (i.e., function $f : X \to |T_{\Sigma}(Y)|$), the function $\mathbf{Val}^{\mathrm{EL}}_{\Sigma,A}(f) : \mathbf{Val}^{\mathrm{EL}}_{\Sigma,A}(Y) \to \mathbf{Val}^{\mathrm{EL}}_{\Sigma,A}(X)$ for every valuation $v : Y \to |A|$ gives the valuation $f;|v^{\sharp}| : X \to |A|$.

For every morphism $\sigma : \langle \Sigma, A \rangle \to \langle \Sigma', A' \rangle$ in $\mathbf{Elts}(\mathbf{Mod}^{\mathrm{EL}})$ its image under $\mathbf{Val}^{\mathrm{EL}}$ is a pair $\langle (\mathbf{Ctxt}^{\mathrm{EL}}_{\sigma})^{op}, \mathbf{val}^{\mathrm{EL}}_{\sigma} \rangle$, where the natural transformation $\mathbf{val}^{\mathrm{EL}}_{\sigma} : \mathbf{Val}^{\mathrm{EL}}_{\Sigma,A} \Rightarrow (\mathbf{Ctxt}^{\mathrm{EL}}_{\sigma})^{op} ; \mathbf{Val}^{\mathrm{EL}}_{\Sigma',A'}$ is given by:

$$\mathbf{val}^{\mathrm{EL}}_{\sigma}(X) \mathrel{\widehat{=}} (\!|\ \_\ |\!)^{X,|A'|}_{\sigma}$$

for every context $X \in |\mathcal{T}_{\mathbf{\Sigma}}|$, where $(\!|\ \_\ |\!)^{X,|A'|}_{\sigma}$ is an appropriate instance of the natural isomorphism described in Sect. 2.

For every signature $\Sigma \in |\mathbf{AlgSig}|$, algebra $A \in |\mathbf{Alg}(\Sigma)|$ and context $X \in |\mathbf{Ctxt}^{\mathrm{EL}}_{\Sigma}|$, the satisfaction relation $A[\_] \models^{\mathrm{EL}}_{\Sigma,X} \_$ is defined by:

$$A[v] \models^{\mathrm{EL}}_{\Sigma,X} t_1 \equiv t_2 \quad \text{iff} \quad (t_1)^v_A = (t_2)^v_A,$$

where $v : X \to |A|$, $t_1 \equiv t_2$ is a Σ-equation with variables from X, and $(t_1)^v_A$ denotes the *interpretation* of the term t extending the valuation v.

The Coherence Condition for EL holds by construction. The Satisfaction Condition is a simple consequence of the fact that for any algebraic signature morphism $\sigma : \Sigma \to \Sigma'$, every S-sorted set of variables X, every Σ'-algebra A' and every valuation $v : X \to |\mathbf{Alg}_{\sigma}(A')|$:

$$v^{\sharp} = (\iota^{\sigma}_X)^{\sharp} ; \mathbf{Alg}_{\sigma}((\!|\, v \,|\!)^{\sharp}_{\sigma}).$$

[2] In both $\mathbf{ISet}_{\sigma}$ and $\mathbf{SSet}_{\sigma}$, for readability reasons, we have used signature morphism σ instead of its *sort function*. We shall use this notational convention in other places as well, without mentioning it. We hope it will not lead to confusion

The Substitution Condition follows directly from the fact, that for every context morphism $f : X \to Y$ in $\mathbf{Ctxt}^{\mathrm{EL}}_{\Sigma}$, every, Σ-algebra A and every valuation $v : Y \to |A|$:

$$(f; |v^{\sharp}|)^{\sharp} = f^{\sharp}; v^{\sharp}.$$

3.3 Category of Abstract Context Institutions

The notion of a morphism between abstract context institutions is a specialization of the corresponding notion for ordinary institutions (cf. [6]). Informally speaking, in typical cases, the existence of a morphism between given abstract context institutions expresses the fact, that one of them (the source of the morphism) "extends" the other.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be abstract context institutions. A morphism $\mathfrak{f} : \mathfrak{C} \to \mathfrak{D}$ will be given by four mappings binding together *signatures*, *formulae*, *models* and *valuations* of $\mathfrak{C}$ and $\mathfrak{D}$.

Before giving the formal definition, let us informally introduce the components of the morphism $\mathfrak{f}$ one-by-one. We shall use the above mentioned "extension" metaphor for $\mathfrak{f}$, and call its source $\mathfrak{C}$ and target $\mathfrak{D}$ "richer" and "poorer" respectively. We would like to stress that this naming convention is based on "typical examples" and as such is purely informal.

Signatures. The signature part of the morphism $\mathfrak{f}$, as in the case of ordinary institutions, will be given by a functor $\mathfrak{f}^{\mathrm{S}} : \mathbf{Sig}^{\mathfrak{C}} \to \mathbf{Sig}^{\mathfrak{D}}$, "extracting" the *poorer* signatures of $\mathfrak{D}$ from the *richer* ones of $\mathfrak{C}$.

Formulae. The "inclusion" of *poorer* formulae of $\mathfrak{D}$ into *richer* formulae of $\mathfrak{C}$ will be given by a natural transformation $\mathfrak{f}^{\mathrm{F}} : \mathfrak{f}^{\mathrm{S}} ; \mathbf{Frm}^{\mathfrak{D}} \Rightarrow \mathbf{Frm}^{\mathfrak{C}}$. Since the formula functors for abstract context institutions "include" context functors, the components of $\mathfrak{f}^{\mathrm{F}}$ have to deal with contexts as well as with formulae.

For an arbitrary signature Σ of the *richer* institution $\mathfrak{C}$, the component $\mathfrak{f}^{\mathrm{F}}_{\Sigma} : \mathbf{Frm}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)} \to \mathbf{Frm}^{\mathfrak{C}}_{\Sigma}$ (as a morphism in $\mathbf{sDgm}(\mathbf{Set})$) will consist of a functor $\mathfrak{f}^{\mathrm{C}}_{\Sigma} : \mathbf{Ctxt}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)} \to \mathbf{Ctxt}^{\mathfrak{C}}_{\Sigma}$ and a natural transformation $\mathbf{frm}(\mathfrak{f}^{\mathrm{F}}_{\Sigma}) : \mathbf{Frm}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)} \Rightarrow \mathfrak{f}^{\mathrm{C}}_{\Sigma} ; \mathbf{Frm}^{\mathfrak{C}}_{\Sigma}$.

Models. The model component of the morphism $\mathfrak{f}$, as in the case of ordinary institutions, will be given by a natural transformation $\mathfrak{f}^{\mathrm{M}} : \mathbf{Mod}^{\mathfrak{C}} \Rightarrow \mathfrak{f}^{\mathrm{S}} ; \mathbf{Mod}^{\mathfrak{D}}$, "extracting" from each model of $\mathfrak{C}$ its "submodel" belonging to the *poorer* institution $\mathfrak{D}$.

Valuations. Similarly as in the case of formulae, the morphism $\mathfrak{f}$ "embeds" valuations of the *poorer* institution $\mathfrak{D}$ into the valuations of the *richer* institution $\mathfrak{C}$. Thus, we shall require existence of a natural transformation $\mathfrak{f}^{\mathrm{V}} : \mathbf{Elts}(\langle \mathfrak{f}^{\mathrm{S}}, \mathfrak{f}^{\mathrm{M}} \rangle) ; \mathbf{Val}^{\mathfrak{D}} \Rightarrow \mathbf{Val}^{\mathfrak{C}}$. For every pair $\langle \Sigma, M \rangle$, where $\Sigma \in |\mathbf{Sig}^{\mathfrak{C}}|$ and $M \in \mathbf{Mod}^{\mathfrak{C}}_{\Sigma}$ (i.e., an object of $\mathbf{Elts}(\mathbf{Mod}\mathfrak{C})$), the component $\mathfrak{f}^{\mathrm{V}}_{\Sigma,M}$ of this transformation is a morphism in the category of diagrams $\mathbf{sDgm}(\mathbf{Set})$. We shall assume (see the *Coherence Condition* below), that the first element of the pair $\mathfrak{f}^{\mathrm{V}}_{\Sigma,M}$ is the functor $(\mathfrak{f}^{\mathrm{C}}_{\Sigma})^{op} : (\mathbf{Ctxt}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)})^{op} \to (\mathbf{Ctxt}^{\mathfrak{C}}_{\Sigma})^{op}$. The second element is a natural transformation $\mathbf{val}(\mathfrak{f}^{\mathrm{V}}_{\Sigma,M}) : \mathbf{Val}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma),\mathfrak{f}^{\mathrm{M}}(M)} \Rightarrow (\mathfrak{f}^{\mathrm{C}}_{\Sigma})^{op} ; \mathbf{Val}^{\mathfrak{C}}_{\Sigma,M}$, defining the "inclusion" between the corresponding sets of valuations of the abstract context institutions $\mathfrak{D}$ and $\mathfrak{C}$.

Satisfaction Condition. The Satisfaction Condition for $\mathfrak{f} : \mathfrak{C} \to \mathfrak{D}$ relates mappings given by the natural transformations $\mathfrak{f}^{\mathrm{F}}$, $\mathfrak{f}^{\mathrm{M}}$ and $\mathfrak{f}^{\mathrm{V}}$ to the semantic consequence relations of $\mathfrak{C}$ and $\mathfrak{D}$.

Informally, it says that for every Σ-model M in $\mathfrak{C}$, a formula of the *poorer* institution $\mathfrak{D}$, is satisfied by a valuation v in the "submodel" $\mathfrak{f}^{\mathrm{M}}_{\Sigma}(M) \in \mathbf{Mod}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)}$ if and only if this formula "embedded" into the set of formulae of the *richer* institution $\mathfrak{C}$ (via $\mathfrak{f}^{\mathrm{F}}$) is satisfied by the image of v wrt. $\mathfrak{f}^{\mathrm{V}}$.

Definition 2. *An abstract context institution morphism* $\mathfrak{f} : \mathfrak{C} \to \mathfrak{D}$ *consists of:*

- *a functor* $\mathfrak{f}^{\mathrm{S}} : \mathbf{Sig}^{\mathfrak{C}} \to \mathbf{Sig}^{\mathfrak{D}}$,
- *a natural transformation* $\mathfrak{f}^{\mathrm{F}} : \mathfrak{f}^{\mathrm{S}} ; \mathbf{Frm}^{\mathfrak{D}} \Rightarrow \mathbf{Frm}^{\mathfrak{C}}$,
- *a natural transformation* $\mathfrak{f}^{\mathrm{M}} : \mathbf{Mod}^{\mathfrak{C}} \Rightarrow (\mathfrak{f}^{\mathrm{S}})^{op} ; \mathbf{Mod}^{\mathfrak{D}}$
- *a natural transformation* $\mathfrak{f}^{\mathrm{V}} : \mathbf{Elts}(\langle \mathfrak{f}^{\mathrm{S}}, \mathfrak{f}^{\mathrm{M}} \rangle) ; \mathbf{Val}^{\mathfrak{D}} \Rightarrow \mathbf{Val}^{\mathfrak{C}}$,

such that the following two conditions are satisfied:

- Coherence Condition*:* $\mathbf{bas}(\mathbf{ind}(\mathbf{Mod}^{\mathfrak{C}}) * \mathfrak{f}^{\mathrm{F}}) = \mathbf{bas}(\mathfrak{f}^{\mathrm{V}}) * (\_)^{op}$
  which says, that for every signature Σ *and model* $M \in \mathbf{Mod}^{\mathfrak{C}}_{\Sigma}$*, the first component of the pair* $\mathfrak{f}^{\mathrm{V}}_{\Sigma,M}$ *is the functor* $(\mathfrak{f}^{\mathrm{C}}_{\Sigma})^{op} : (\mathbf{Ctxt}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)})^{op} \to (\mathbf{Ctxt}^{\mathfrak{C}}_{\Sigma})^{op}$,[3] *where* $\mathfrak{f}^{\mathrm{C}} \mathrel{\widehat{=}} \mathbf{bas}(\mathfrak{f}^{\mathrm{F}})$,
- Satisfaction Condition*: for every signature* Σ *in* $\mathfrak{C}$*, and every: model* $M \in \mathbf{Mod}^{\mathfrak{C}}_{\Sigma}$*, context* $\Gamma \in |\mathbf{Ctxt}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)}|$*, formula* $\psi \in \mathbf{Frm}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma)}(\Gamma)$ *and valuation* v *from the set* $\mathbf{Val}^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma),\mathfrak{f}^{\mathrm{M}}(M)}(\Gamma)$*:*

  $$M[\mathbf{val}(\mathfrak{f}^{\mathrm{V}}_{\Sigma,M})(\Gamma)(v)] \models^{\mathfrak{C}}_{\Sigma,\mathfrak{f}^{\mathrm{C}}_{\Sigma}(\Gamma)} \mathbf{frm}(\mathfrak{f}^{\mathrm{F}}_{\Sigma})(\Gamma)(\psi) \quad \textit{iff} \quad \mathfrak{f}^{\mathrm{M}}_{\Sigma}(M)[v] \models^{\mathfrak{D}}_{\mathfrak{f}^{\mathrm{S}}(\Sigma),\Gamma} \psi.$$

The composition of abstract context institution morphisms $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ and $\mathfrak{g} : \mathfrak{B} \to \mathfrak{C}$ is defined in a straightforward way:

- $(\mathfrak{f} ; \mathfrak{g})^{\mathrm{S}} \mathrel{\widehat{=}} \mathfrak{f}^{\mathrm{S}} ; \mathfrak{g}^{\mathrm{S}}$,
- $(\mathfrak{f} ; \mathfrak{g})^{\mathrm{F}} \mathrel{\widehat{=}} (\mathfrak{f}^{\mathrm{S}} * \mathfrak{g}^{\mathrm{F}}) ; \mathfrak{f}^{\mathrm{F}}$,
- $(\mathfrak{f} ; \mathfrak{g})^{\mathrm{M}} \mathrel{\widehat{=}} \mathfrak{f}^{\mathrm{M}} ; (\mathfrak{f}^{\mathrm{S}} * \mathfrak{g}^{\mathrm{M}})$,
- $(\mathfrak{f} ; \mathfrak{g})^{\mathrm{V}} \mathrel{\widehat{=}} (\mathbf{Elts}(\langle \mathfrak{f}^{\mathrm{S}}, \mathfrak{f}^{\mathrm{M}} \rangle) * \mathfrak{g}^{\mathrm{V}}) ; \mathfrak{f}^{\mathrm{V}}$.

Abstract context institutions and their morphisms form a category which we shall denote by $\mathfrak{AbsConIns}$.

[3] Note the similarity of this condition and the Coherence Condition from the definition of abstract context institution.

Due to the lack of space, instead of giving a "direct" example, let us only briefly mention, how we can, using abstract context institution morphisms, generalize the construction of the institution of *abstract Hoare logic*, described in [12].

As the basis for the construction, we take a morphism $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$, between abstract context institutions of the *logic of assertions* $\mathfrak{A}$, and the *logic of Boolean expressions* $\mathfrak{B}$. Using context morphisms of $\mathfrak{B}$ as atomic commands, we define an abstract imperative programming language, and give its "institutional" semantics. Then, we can construct an (ordinary) institution $\mathcal{H}(\mathfrak{f})$, of *Hoare logic for* $\mathfrak{f}$, taking appropriately defined *abstract Hoare triples* as sentences.

For the institution $\mathcal{H}(\mathfrak{f})$ we can define Hoare-like inference system, and using only the structural properties of $\mathfrak{A}$, $\mathfrak{B}$, and the morphism $\mathfrak{f}$, we can show, that this system is sound and Cook-complete (cf. [14], Chapt. 6).

4 Presentations

4.1 Metasignatures and Metastructures

The aim of this section is to introduce some auxiliary notions, needed for the definition of *presentation*. *Metasignatures* and *metastructures* will be used for "presenting" the syntactic and semantic aspects of a logical system respectively.[4]

Definition 3. *A* metasignature *is a six-tuple* $L = \langle S, \Omega, \Pi, V, C, Q \rangle$*, such that* $\langle S, \Omega, \Pi \rangle$ *is a relational signature,* $V \subseteq S$*,* C *is a family of sets indexed by natural numbers, and* Q *is a set.*

The intended interpretation of the respective components will become clear later. For the moment, we can informally say, that metasignatures are just relational signatures enriched by symbols of *logical connectives*, (the family C), *quantifier symbols* (the set Q), and having a distinguished subset of sort names (the set V), for which we want to talk about *variables*.

Definition 4. *A* metasignature morphism $\ell : L \to L'$ *is a triple* $\ell = \langle \ell_r, \ell_c, \ell_q \rangle$*, such that:*

- $\ell_r : \langle S, \Omega, \Pi \rangle \to \langle S', \Omega', \Pi' \rangle$ *is a relational signature morphism s.t.* $\ell_r[V]$ *is a subset of* V'*,*
- $\ell_c = \langle \ell_{c,k} : C_k \to C'_k \mid k \in \mathbb{N} \rangle$ *and*
- $\ell_q : Q \to Q'$ *is a function.*

It is easy to see, that metasignatures and their morphisms (with an obvious definition of composition) constitute a category. We shall denote it by **MSig**.

For every metasignature $L = \langle S, \Omega, \Pi, V, C, Q \rangle$, by $\mathbf{Syn}(L)$ we shall denote an algebraic signature defined as follows:

[4] The notions of *metasignature* and *metastructure* defined in this paper are slightly different from the analogous notions presented in [12]. The main difference is a separation of *predicate symbols* from *operation symbols* in metasignatures.

– *sorts*: $S \uplus \{\star\}$[5]
– *operation symbols*: $\Omega^{\mathrm{Syn}} \cup \Pi^{\mathrm{Syn}} \cup C^{\mathrm{Syn}}$, where:
$$\Omega^{\mathrm{Syn}}_{t_1 \ldots t_n \to t_0} \mathrel{\widehat{=}} \begin{cases} \Omega_{s_1 \ldots s_n \to s_0} & \text{if } t_i = T(s_i) \text{ for } i = 0 \ldots n \\ \emptyset & \text{otherwise} \end{cases}$$
$$\Pi^{\mathrm{Syn}}_{t_1 \ldots t_n \to t_0} \mathrel{\widehat{=}} \begin{cases} \Pi_{s_1 \ldots s_n} & \text{if } t_i = T(s_i) \text{ for } i = 1 \ldots n \text{ and } t_0 = \mathbb{B} \\ \emptyset & \text{otherwise} \end{cases}$$
$$C^{\mathrm{Syn}}_{t_1 \ldots t_n \to t_0} \mathrel{\widehat{=}} \begin{cases} C_n & \text{if } t_i = \mathbb{B} \text{ for } i = 0 \ldots n \\ \emptyset & \text{otherwise} \end{cases}.$$

As it is easy to check, the above construction can be extended to a functor $\mathbf{Syn} : \mathbf{MSig} \to \mathbf{AlgSig}$. In what follows, we shall also use two "sub-functors" of **Syn**—$\mathbf{Atm} : \mathbf{MSig} \to \mathbf{AlgSig}$ and $\mathbf{Trm} : \mathbf{MSig} \to \mathbf{AlgSig}$, such that:

– $\mathbf{Atm}(\langle S, \Omega, \Pi, V, C, Q \rangle) \mathrel{\widehat{=}} \langle T(S) \cup \{\mathbb{B}\}, \Omega^{\mathrm{Syn}} \cup \Pi^{\mathrm{Syn}} \rangle$,
– $\mathbf{Trm}(\langle S, \Omega, \Pi, V, C, Q \rangle) \mathrel{\widehat{=}} \langle T(S), \langle \Omega^{\mathrm{Syn}}_{w \to t} \mid w \in T(S)^* \wedge t \in T(S) \rangle \rangle$,

The morphisms $\mathbf{Atm}_\ell$ and $\mathbf{Trm}_\ell$ are given by the respective components of the morphism $\mathbf{Syn}_\ell$. For every metasignature L, the signatures $\mathbf{Atm}(L)$ and $\mathbf{Trm}(L)$ are *sub-signatures* of $\mathbf{Syn}(L)$—hence the informal term "sub-functor".

Definition 5. *An* L-metastructure $\mathcal{A}$ *consists of:*

– *a* $\mathbf{Syn}(L)$*-algebra* A*,*
– *a* $T(V)$*-indexed set* $V_{\mathcal{A}}$*, such that for every* $s \in V$ $(V_{\mathcal{A}})_{T(s)} \subseteq |A|_{T(s)}$*,*
– *a set* $D_{\mathcal{A}}$*, such that* $D_{\mathcal{A}} \subseteq |A|_{\mathbb{B}}$*,*
– *for every symbol* $q \in Q$*, a partial function* $q_{\mathcal{A}} : \mathcal{P}(|A|_{\mathbb{B}}) \rightharpoonup |A|_{\mathbb{B}}$*.*

The set $|\mathcal{A}| \mathrel{\widehat{=}} |A|$ *will be called the* carrier *of the metastructure* $\mathcal{A}$*, the subset* $V_{\mathcal{A}}$*—its* set of values*, and the set* $D_{\mathcal{A}}$*—its* set of designated elements*. The set* $|\mathcal{A}|_{\mathbb{B}}$*, corresponding to the distinguished sort* $\mathbb{B}$ *of the "syntactic" signature* $\mathbf{Syn}(L)$*, will play the rôle of the set of* logical values*.*

Essentially, L-metastructures are just many-sorted algebras enriched by *generalized operations*—the partial functions corresponding to the symbols from the set Q. In what follows, these operations will be used for giving the semantics of *quantifiers*.[6]

Definition 6. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* L*-metastructures. A* metastructure morphism $h : \mathcal{A} \to \mathcal{A}'$ *is a* $\mathbf{Syn}(L)$*-homomorphism* $h : A \to A'$*, satisfying the following conditions:*

– *for every symbol* $q \in Q$*, whenever* B *belongs to the domain of the generalized operation* $q_{\mathcal{A}}$*, then* $h[B]$ *(the image of* B *wrt.* h*) belongs to the domain of* $q_{\mathcal{A}'}$ *and* $h(q_{\mathcal{A}}(B)) = q_{\mathcal{A}'}(h[B])$*,*

[5] The injection of $s \in S$ and $\star$ into the disjoint union $S \uplus \{\star\}$ will be denoted by $T(s)$ and $\mathbb{B}$ respectively. By $T(S)$ we shall denote the set $\{ T(s) \mid s \in S \}$.
[6] A similar idea, although in a slightly different context, appeared in the work of H. Rasiowa and R. Sikorski on *algebraization of logic* [16, 15].

- $h[V_{\mathcal{A}}] \subseteq V_{\mathcal{A}'}$,
- $h[D_{\mathcal{A}}] \subseteq D_{\mathcal{A}'}$.

For every metasignature L, the class of all L-metastructures together with their morphisms form a category, which we shall denote by $\mathbf{MStr}(L)$. The composition of morphisms in $\mathbf{MStr}(L)$ is defined as the composition of $\mathbf{Syn}(L)$-homomorphisms in $\mathbf{Alg}(\mathbf{Syn}(L))$—the category of $\mathbf{Syn}(L)$-algebras.

Using the functor $\mathbf{Syn}$, for any metasignature morphism $\ell : L \to L'$, we can define a *ℓ-reduct functor* $\mathbf{MStr}_\ell : \mathbf{MStr}(L') \to \mathbf{MStr}(L)$. This construction extends to an indexed category $\mathbf{MStr} : \mathbf{MSig}^{op} \to \mathbf{Cat}$.

Below, using notions of metasignature and metastructure, we shall define *interpretation structures*, which will play a fundamental rôle in the definition of *presentation*.

4.2 Interpretation Structures

Let us assume, that there is a class of (abstract) objects, which we shall denote by $\mathcal{M}$, and whose elements will be called *models*. An *interpretation function for* $\mathcal{M}$ is a function, which for every element of $\mathcal{M}$ returns a metastructure over a fixed "*metalanguage*" metasignature (the same for the whole class $\mathcal{M}$). More formally:

Definition 7. *An* interpretation structure *is a triple* $\langle L, \mathcal{M}, Int \rangle$, *consisting of:*

- *a* metalanguage signature $L \in |\mathbf{MSig}|$
- *a class of* models $\mathcal{M} \in |\mathbf{Class}|$,
- *an* interpretation function *(functor)* $Int : \mathcal{M} \to \mathbf{MStr}(L)$.

Definition 8. *A triple* $\langle \ell, m, int \rangle$ *is an* interpretation structure morphism *from* $\langle L, \mathcal{M}, Int \rangle$ *to* $\langle L', \mathcal{M}', Int' \rangle$ *iff:*

- $\ell : L \to L'$ *is a metasignature morphism,*
- $m : \mathcal{M}' \to \mathcal{M}$ *is a function,*
- $int : m\,; Int \Rightarrow Int'\,; \mathbf{MStr}_\ell$ *is a natural transformation.*

Since $\mathcal{M}'$ is a discrete category (a class) the natural transformation *int* is simply an $\mathcal{M}'$-indexed family of metastructure morphisms, such that for every $M' \in \mathcal{M}'$:

$$int_{M'} : Int(m(M')) \to \mathbf{MStr}_\ell(Int'(M')).$$

The composition of $\langle \ell, m, int \rangle : IS_1 \to IS_2$ and $\langle \ell', m', int' \rangle : IS_2 \to IS_3$ is defined by:

$$\langle \ell, m, int \rangle\,; \langle \ell', m', int' \rangle \mathrel{\widehat{=}} \langle\ \ell\,; \ell',\ m'; m,\ (m' {*} int)\,; (int' {*} \mathbf{MStr}_\ell)\ \rangle.$$

As it is easy to check, interpretation structures and their morphisms constitute a category. We shall denote it by $\mathbf{IntStr}$.

4.3 Presentations

A *presentation* is an arbitrary functor into the category of interpretation structures:

$$\mathfrak{P} : \mathbf{Sig}^{\mathfrak{P}} \to \mathbf{IntStr}.$$

We shall call $\mathbf{Sig}^{\mathfrak{P}}$ the *category of signatures* of the presentation $\mathfrak{P}$.

For every morphism $\sigma : \Sigma \to \Sigma'$ in $\mathbf{Sig}^{\mathfrak{P}}$, let $\mathfrak{P}(\Sigma) = \langle L_\Sigma, \mathcal{M}_\Sigma, Int_\Sigma \rangle$ and $\mathfrak{P}(\sigma) = \langle \ell_\sigma, m_\sigma, int_\sigma \rangle$. The presentation $\mathfrak{P}$ induces two functors: a *metalanguage functor* and a *model functor*, such that:

- *metalanguage functor* $\mathbf{Lan}^{\mathfrak{P}} : \mathbf{Sig}^{\mathfrak{P}} \to \mathbf{MSig}$
  - $\mathbf{Lan}^{\mathfrak{P}}(\Sigma) \mathrel{\widehat{=}} L_\Sigma$,
  - $\mathbf{Lan}^{\mathfrak{P}}(\sigma) \mathrel{\widehat{=}} \ell_\sigma$;
- *model functor* $\mathbf{Mod}^{\mathfrak{P}} : (\mathbf{Sig}^{\mathfrak{P}})^{op} \to \mathbf{Class}$
  - $\mathbf{Mod}^{\mathfrak{P}}(\Sigma) \mathrel{\widehat{=}} \mathcal{M}_\Sigma$,
  - $\mathbf{Mod}^{\mathfrak{P}}(\sigma) \mathrel{\widehat{=}} m_\sigma$.

In what follows, by $\mathbf{Syn}^{\mathfrak{P}}$, $\mathbf{Atm}^{\mathfrak{P}}$ and $\mathbf{Trm}^{\mathfrak{P}}$, we shall denote the compositions: $\mathbf{Lan}^{\mathfrak{P}} ; \mathbf{Syn}$, $\mathbf{Lan}^{\mathfrak{P}} ; \mathbf{Atm}$ and $\mathbf{Lan}^{\mathfrak{P}} ; \mathbf{Trm}$ (see Sect. 4.1). We shall call $\mathbf{Atm}^{\mathfrak{P}}$ and $\mathbf{Trm}^{\mathfrak{P}}$ the *atomic formula functor* and the *term functor* for $\mathfrak{P}$ respectively. The meaning of these names shall be clarified in Sect. 4.4, where we describe a construction of an abstract context institution out of a (logical) presentation. Before going any further let us give some examples of presentations.

Model presentations. Let us start with a very simple one. Since (abstract context) institutions advocate a *model-centric* view of logic, we shall show how to "present" a *model part* of a logical system. We shall do this for the case of (many-sorted) algebras.

Let $\mathcal{ALG} : \mathbf{AlgSig} \to \mathbf{IntStr}$ be a presentation such that:

- For every algebraic signature $\Sigma = \langle S, \Omega \rangle$, the metalanguage signature for $\mathcal{ALG}(\Sigma)$ is defined as $\mathbf{Lan}^{\mathcal{ALG}}(\Sigma) \mathrel{\widehat{=}} \langle S, \Omega, \varnothing, \emptyset, \varnothing, \emptyset \rangle$.[7] The class of models $\mathbf{Mod}^{\mathcal{ALG}}(\Sigma)$ is the class of all Σ-algebras. The interpretation function maps every Σ-algebra A to a $\mathbf{Lan}^{\mathcal{ALG}}(\Sigma)$-metastructure, extending A by a Boolean sort, defined as: $|Int^{\mathcal{ALG}}_\Sigma(A)|_{\mathbb{B}} \mathrel{\widehat{=}} \{ \mathtt{tt}, \mathtt{ff} \}$, and taking $\{ \mathtt{tt} \}$ as its set of designated elements.
- For every algebraic signature morphism $\sigma : \Sigma \to \Sigma'$, the morphism $\mathcal{ALG}_\sigma$ consists of: an obvious metasignature morphism $\mathbf{Lan}^{\mathcal{ALG}}_\sigma$ "induced" by σ, the $\mathbf{Syn}^{\mathcal{ALG}}_\sigma$-reduct operation, and a trivial natural transformation, whose all components are identities.

In a very similar way one can define a model presentation for the case of *relational structures* $\mathcal{STR} : \mathbf{RelSig} \to \mathbf{IntStr}$. The only interesting difference is that the interpretation functions have to map the predicates occurring in relational structures to Boolean-valued operations in metastructures.

[7] The symbol $\varnothing$ denotes an appropriately indexed family of empty sets.

Equational logic. Let us now describe a presentation $\mathcal{EL}$, for the (many-sorted) equational logic (an abstract context institution for it has been introduced in Sect. 3.2).

The category of signatures $\mathbf{Sig}^{\mathcal{EL}}$ is the category of algebraic signatures **AlgSig**. The metalanguage functor $\mathbf{Lan}^{\mathcal{EL}} : \mathbf{AlgSig} \to \mathbf{MSig}$, and the model functor $\mathbf{Mod}^{\mathcal{EL}} : \mathbf{AlgSig}^{op} \to \mathbf{Class}$ are defined as follows:

- $\mathbf{Lan}^{\mathcal{EL}}(\langle S, \Omega \rangle) \mathrel{\widehat{=}} \langle S, \Omega, \Pi, S, C, \emptyset \rangle$, where $\Pi_{ss} \mathrel{\widehat{=}} \{\equiv\}$ for $s \in S$, and all other elements of Π, and all C_n for $n \geq 0$ are empty.
- for every $\sigma : \Sigma \to \Sigma'$ in **AlgSig**, the morphism $\mathbf{Lan}^{\mathcal{EL}}_{\sigma} : \mathbf{Lan}^{\mathcal{EL}}(\Sigma) \to \mathbf{Lan}^{\mathcal{EL}}(\Sigma')$ is defined as σ, for the symbols coming from Σ, and as the identity for the *equations* from Π_{ss}, for $s \in S$.
- as the class $\mathbf{Mod}^{\mathcal{EL}}_{\Sigma}$, we take the class of all Σ-algebras $\mathbf{Alg}[\Sigma]$. For every $\sigma : \Sigma \to \Sigma'$ in **AlgSig**, a function $\mathbf{Mod}^{\mathcal{EL}}_{\sigma} : \mathbf{Mod}^{\mathcal{EL}}_{\Sigma'} \to \mathbf{Mod}^{\mathcal{EL}}_{\Sigma}$ is the algebraic σ-reduct operation.

Let $\Sigma = \langle S, \Omega \rangle$ be an algebraic signature. For every Σ-model (algebra) M, we shall now define the metamodel $Int_{\Sigma}(M)$. Let A_M be a $\mathbf{Syn}^{\mathcal{EL}}(\Sigma)$-algebra such that:

- for every sort $s \in S$, let $|A_M|_{T(s)} \mathrel{\widehat{=}} |M|_s$,
- $|A_M|_{\mathbb{B}} \mathrel{\widehat{=}} \{\mathtt{tt}, \mathtt{ff}\}$,
- for every $\omega : s_1 \dots s_n \to s_0$, $\omega_{A_M} \mathrel{\widehat{=}} \omega_M$,
- for every $s \in S$ and $a_1, a_2 \in |A_M|_{T(s)}$,

$$\equiv_{A_M} (a_1, a_2) \mathrel{\widehat{=}} \begin{cases} \mathtt{tt} \text{ if } a_1 = a_2 \\ \mathtt{ff} \text{ otherwise} \end{cases}$$

For every $s \in S$, as the *set of values* $|V^{Int_{\Sigma}(M)}|_{T(s)}$ for the sort $T(s)$, let us take the whole set $|A_M|_{T(s)}$, and as the set of *designated elements* $D^{Int_{\Sigma}(M)}$—the singleton set $\{\mathtt{tt}\}$.

For every algebraic signature morphism $\sigma : \Sigma \to \Sigma'$ and every algebra $M' \in \mathbf{Mod}^{\mathcal{EL}}_{\Sigma'}$, let $int^{\sigma}_{M'}$ be the identity morphism.

Partial First-Order Logic. As an even more interesting example, we shall describe a presentation $\mathcal{PFOL}$ for the partial first-order logic (cf. [4]), adding two interesting features: *quantifiers* and *partiality* to the picture.

The category of signatures for $\mathcal{PFOL}$ is the category of *partial relational signatures* **PRelSig**, whose objects are quadruples $\langle S, \Omega, \mathrm{p}\Omega, \Pi \rangle$, such that $\langle S, \Omega, \Pi \rangle$ is a relational signature, and $\mathrm{p}\Omega$ is an $S^* \times S$-indexed set of *partial operation names*. Morphisms in **PRelSig** are defined in the expected way (see [4]).

The metalanguage functor $\mathbf{Lan}^{\mathcal{PFOL}} : \mathbf{PRelSig} \to \mathbf{MSig}$ is defined as follows:

- $\mathbf{Lan}^{\mathcal{PFOL}}(\langle S, \Omega, \mathrm{p}\Omega, \Pi \rangle) \mathrel{\widehat{=}} \langle S, \Omega \cup \mathrm{p}\Omega, \Pi(\stackrel{e}{\equiv}), S, C, \{\forall\} \rangle$, where the family $\Pi(\stackrel{e}{\equiv})$ extends Π by adding a symbol $\stackrel{e}{\equiv}$ to every Π_{ss}, for $s \in S$,[8] $C_1 \mathrel{\widehat{=}} \{\neg\}$, $C_2 \mathrel{\widehat{=}} \{\wedge\}$, and the sets: C_0 and C_n for $n \geq 3$ are empty,

[8] We assume here, that the symbol $\stackrel{e}{\equiv}$ did not belong to any of the sets Π_{ss}.

– for every partial relational signature morphism $\sigma : \Sigma \to \Sigma'$, the morphism $\mathbf{Lan}^{\mathcal{PFOL}}_\sigma : \mathbf{Lan}^{\mathcal{PFOL}}(\Sigma) \to \mathbf{Lan}^{\mathcal{PFOL}}(\Sigma')$ is given by σ, for symbols coming from Σ, and is defined as the identity for the symbols of *existentional equality*, *connectives*, and *quantifiers*—i.e., for the symbols: $\stackrel{e}{\equiv}$, $\neg$, $\wedge$ and $\forall$.

The functor $\mathbf{Mod}^{\mathcal{PFOL}} : \mathbf{PRelSig}^{op} \to \mathbf{Class}$, for every partial relational signature Σ, returns the class of all *partial relational Σ-structures* (defined in an obvious way). For every $\sigma : \Sigma \to \Sigma'$ in **PRelSig**, the function $\mathbf{Mod}^{\mathcal{PFOL}}_\sigma$ is the corresponding σ-reduct operation.

Let us now define the interpretation function for $\Sigma = \langle S, \Omega, \mathrm{p}\Omega, \Pi \rangle$. Let M be an arbitrary partial relational Σ-structure. We have to define the metastructure $Int_\Sigma(M)$. Let A_M be a $\mathbf{Syn}^{\mathcal{PFOL}}(\Sigma)$-algebra, such that:

– for every $s \in S$, the set $|A_M|_{T(s)}$, is obtained from $|M|_s$, by adding an extra "undefined" element (different from all the elements of $|M|_s$), which shall always denote by $\perp$,
– $|A_M|_{\mathbb{B}} \mathrel{\widehat{=}} \{\mathtt{tt}, \mathtt{ff}\}$,
– for every $\omega : s_1 \ldots s_n \to s_0$ and $a_i \in |A_M|_{T(s_i)}$, $i = 0 \ldots n$

$$\omega_{A_M}(a_1, \ldots, a_n) \mathrel{\widehat{=}} \begin{cases} \omega_M(a_1, \ldots, a_n) & \text{if } \langle a_1, \ldots, a_n \rangle \in \mathbf{dom}(\omega_M) \\ \perp & \text{otherwise} \end{cases}$$

– for every $\pi : s_1 \ldots s_n$ and $a_i \in |A_M|_{T(s_i)}$, $i = 1 \ldots n$

$$\pi_{A_M}(a_1, \ldots, a_n) \mathrel{\widehat{=}} \begin{cases} \mathtt{tt} & \text{if } \langle a_1, \ldots, a_n \rangle \in \pi_M \\ \mathtt{ff} & \text{otherwise} \end{cases}$$

– for every sort $s \in S$, and every $a_1, a_2 \in |A_M|_{T(s)}$

$$\stackrel{e}{\equiv}_{A_M}(a_1, a_2) \mathrel{\widehat{=}} \begin{cases} \mathtt{tt} & \text{if } a_1 = a_2 \text{ and } a_i \neq \perp \text{ for } i = 1, 2 \\ \mathtt{ff} & \text{otherwise} \end{cases}$$

– for every $b, b' \in |A_M|_{\mathbb{B}}$

$$\neg_{A_M}(b) \mathrel{\widehat{=}} \begin{cases} \mathtt{tt} & \text{if } b = \mathtt{ff} \\ \mathtt{ff} & \text{otherwise} \end{cases} \qquad \wedge_{A_M}(b, b') \mathrel{\widehat{=}} \begin{cases} \mathtt{tt} & \text{if } b = \mathtt{tt} = b' \\ \mathtt{ff} & \text{otherwise} \end{cases}$$

For every $s \in S$, as the *set of values* $|V^{Int_\Sigma(M)}|_{T(s)}$, let us take the set $|M|_s$. In other words, the values are all elements of the carrier of M. The "undefined elements" $\perp$—are not values. As the *set of designated elements* $D^{Int_\Sigma(M)}$, as in the algebraic case, let us take the set $\{\mathtt{tt}\}$.

We still have to define the generalized operation $\forall_{Int_\Sigma(M)}$, interpreting the $\forall$ quantifier. Let us take, for every $B \subseteq |A_M|_{\mathbb{B}}$:

$$\forall_{Int_\Sigma(M)}(B) \mathrel{\widehat{=}} \begin{cases} \mathtt{ff} & \text{if } \mathtt{ff} \in B \\ \mathtt{tt} & \text{otherwise.} \end{cases}$$

For every signature morphism $\sigma : \Sigma \to \Sigma'$ in **PRelSig**, and every model $M' \in \mathbf{Mod}^{\mathcal{PFOL}}_{\Sigma'}$, let us take the identity as the morphism:

$$int^\sigma_{M'} : Int_\Sigma(\mathbf{Mod}^{\mathcal{PFOL}}_\sigma(M')) \to \mathbf{MStr}_{\mathbf{Lan}^{\mathcal{PFOL}}(\sigma)}(Int_{\Sigma'}(M')).$$

Boolean Presentations. In the case of *model functor presentations*, their main nontrivial part was a model functor, For *Boolean presentations* it will be an *algebra of truth values*.

By a *signature of truth values* we shall mean an arbitrary metasignature of the form $\langle \emptyset, \varnothing, \varnothing, \emptyset, C, Q \rangle$ (in short, we shall denote it by $\langle C, Q \rangle$). A metastructure B over $\langle C, Q \rangle$ will be called an *algebra of truth values*, if all the generalized operations $q_B : \mathcal{P}(|B|) \rightharpoonup |B|$, for $q \in Q$, are total.

For every $\langle C, Q \rangle$-algebra of truth values B, let us define the corresponding presentation $\mathcal{B}(C, Q)$. Its category of signatures is an arbitrary discrete, singleton category, whose only object shall be denoted by „$\heartsuit$". The other components of $\mathcal{B}(C, Q)$ are defined as follows:

- $\mathbf{Lan}_{\heartsuit}^{\mathcal{B}(C,Q)} \mathrel{\widehat{=}} \langle \emptyset, \varnothing, \varnothing, \emptyset, C, Q \rangle$,
- $\mathbf{Mod}_{\heartsuit}^{\mathcal{B}(C,Q)} \mathrel{\widehat{=}} \{\spadesuit\}$,[9]
- $\mathit{Int}_{\heartsuit}^{\mathcal{B}(C,Q)} \mathrel{\widehat{=}} B$.

Presentations of the above form shall be called *Boolean*.

The "canonical" example of a Boolean presentation is a presentation for the *classical truth values*. It corresponds to the $\langle \varnothing, \emptyset \rangle$-algebra of truth values `Bool`, whose carrier is a set $\{ \mathtt{tt}, \mathtt{ff} \}$, and $\{ \mathtt{tt} \}$ is the set of designated elements. We shall denote this presentation by $\mathcal{B}ool$. Presentations corresponding to "enrichments" of the algebra `Bool` by connectives C and quantifiers Q (with a fixed semantics) shall be denoted by $\mathcal{B}ool(C, Q)$. For example, a Boolean presentation for the classical truth values with negation, conjunction and the universal quantifier (the corresponding generalized operation was defined in the previous section—example of $\mathcal{PFOL}$) shall be denoted by $\mathcal{B}ool(\{\neg, \wedge\}, \{\forall\})$.

In what follows a presentation $\mathfrak{P}$ will be called *logical* iff:

- for every signature Σ in $\mathbf{Sig}^{\mathfrak{P}}$, and every model $M \in \mathbf{Mod}_{\Sigma}^{\mathfrak{P}}$, all the generalized operations in the metastructure $\mathit{Int}_{\Sigma}(M)$ are *total*,
- for every morphism $\sigma : \langle \Sigma, M \rangle \to \langle \Sigma', M' \rangle$ in $\mathbf{Elts}(\mathbf{Mod}^{\mathfrak{P}})$, the morphism $\mathit{int}_{M'}^{\sigma} : \mathit{Int}_{\Sigma}(M) \to \mathbf{MStr}_{\mathbf{Lan}_{\sigma}^{\mathfrak{P}}}(\mathit{Int}_{\Sigma'}(M'))$ satisfies the following conditions:
  - for every element $b \in |\mathit{Int}_{\Sigma}(M)|_{\mathbb{B}}$, whenever $\mathit{int}_{M'}^{\sigma}(b)$ is a designated element in $\mathbf{MStr}_{\mathbf{Lan}_{\sigma}^{\mathfrak{P}}}(\mathit{Int}_{\Sigma'}(M'))$, then also b is a designated element in $\mathit{Int}_{\Sigma}(M)$,[10]
  - the morphism $\mathit{int}_{M'}^{\sigma}$ *surjectively* maps the set of values of the metastructure $\mathit{Int}_{\Sigma}(M)$ onto the set of values of $\mathbf{MStr}_{\mathbf{Lan}_{\sigma}^{\mathfrak{P}}}(\mathit{Int}_{\Sigma'}(M'))$.

In an obvious way all the examples of presentations given above are logical. A very important feature of logical presentations is that they *generate* abstract context institutions.

[9] The only element of the $\heartsuit$-model class of $\mathcal{B}(C, Q)$ shall be denoted by „$\spadesuit$" (similarly, as the only signature, which was denoted by „$\heartsuit$").

[10] In other words, $\mathit{int}_{M'}^{\sigma}$ has to both *preserve* (as every metastructure morphism), and *reflect* the designated elements.

4.4 From Presentations to Abstract Context Institutions

Below, we shall sketch a construction, which for an arbitrary presentation $\mathfrak{P}$ yields an abstract context institution $\mathcal{I}(\mathfrak{P})$.

Signatures, Contexts and Formulae. As the category of signatures for $\mathcal{I}(\mathfrak{P})$, we take the category $\mathbf{Sig}^{\mathfrak{P}}$. For an arbitrary signature $\Sigma \in |\mathbf{Sig}^{\mathfrak{P}}|$, let $\mathbf{Lan}^{\mathfrak{P}}(\Sigma) = \langle S, \Omega, \Pi, V, C, Q \rangle$.

For every signature Σ in $\mathbf{Sig}^{\mathfrak{P}}$, as $\mathbf{Ctxt}_{\Sigma}^{\mathcal{I}(\mathfrak{P})}$, we take the full subcategory of the category of substitutions $\mathcal{T}_{\mathbf{Trm}^{\mathfrak{P}}(\mathbf{\Sigma})}$ generated by (S-sorted) sets of *variables* X, s.t. $X_{T(s)} = \emptyset$, for $s \notin V$. This restriction corresponds to the intuitive rôle of the subset $V \subseteq S$, as the set of sorts for which we want to talk about variables. The morphism part of the context functor $\mathbf{Ctxt}^{\mathcal{I}(\mathfrak{P})}$ is given by the functor $\mathcal{T}_{\mathbf{Trm}^{\mathfrak{P}}_{-}}$.

To define, for any Σ-context X, the set of formulae $\mathbf{Frm}_{\Sigma}^{\mathcal{I}(\mathfrak{P})}(X)$, we use (almost) the usual first-order syntax approach. First, we take the set $|T_{\mathbf{Atm}^{\mathfrak{P}}(\Sigma)}(X)|_{\mathbb{B}}$ as the set of *atomic formulae* $A_{\Sigma}(X)$. Then, we close $A_{\Sigma}(X)$ wrt. the connectives and quantifier symbols, obtaining the set $F_{\Sigma}(X)$ of *Σ-pre-formulae with variables from X*. To avoid working with equivalence classes (wrt. α-conversion) of pre-formulae, we use the approach of [17]. Eventually, we define the set $\mathbf{Frm}_{\Sigma}^{\mathcal{I}(\mathfrak{P})}(X)$ as the set of pre-formulae,"normalized" wrt. suitably defined notion of syntactic substitution.

The syntax translation along signature morphisms is defined in the standard way, yielding the formula functor $\mathbf{Frm}^{\mathcal{I}(\mathfrak{P})} : \mathbf{Sig}^{\mathfrak{P}} \to \mathbf{sDgm}(\mathbf{Set})$.

Models and Valuations. As the model functor $\mathbf{Mod}^{\mathcal{I}(\mathfrak{P})}$, we take the model functor for the presentation $\mathfrak{P}$ (i.e., $\mathbf{Mod}^{\mathfrak{P}}$—see Sect. 4.3).

For an arbitrary $\Sigma \in |\mathbf{Sig}^{\mathfrak{P}}|$, let $\mathfrak{P}(\Sigma) = \langle \mathbf{Lan}^{\mathfrak{P}}(\Sigma), \mathbf{Mod}^{\mathfrak{P}}(\Sigma), Int_{\Sigma} \rangle$. Let $X \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{I}(\mathfrak{P})}|$ be an arbitrary Σ-context. As the set $\mathbf{Val}_{\Sigma,M}^{\mathcal{I}(\mathcal{C})}(X)$ we shall take the set of all $T(S)$-indexed functions of the form $v : X \to \langle |Int_{\Sigma}(M)|_{\tau} \mid \tau \in T(S) \rangle$, where $|Int_{\Sigma}(M)|$ is the carrier of the metastructure $Int_{\Sigma}(M)$.

For every morphism $\xi : X \to Y$ in $\mathbf{Ctxt}_{\Sigma}^{\mathcal{I}(\mathfrak{P})}$, the function $\mathbf{Val}_{\Sigma,M}^{\mathcal{I}(\mathfrak{P})}(\xi) : \mathbf{Val}_{\Sigma,M}^{\mathcal{I}(\mathfrak{P})}(Y) \to \mathbf{Val}_{\Sigma,M}^{\mathcal{I}(\mathfrak{P})}(X)$ is defined in the standard way—as in the "algebraic" case.[11] Also the translation of valuations "along" signature morphisms is defined in a similar "algebraic" way.

Satisfaction Relation. The definition of the satisfaction relation for $\mathcal{I}(\mathfrak{P})$ is based upon a suitable notion of a *semantic interpretation of formulae*. For every: signature $\Sigma \in |\mathbf{Sig}^{\mathcal{I}(\mathfrak{P})}|$, model $M \in \mathbf{Mod}^{\mathcal{I}(\mathcal{C})}(\Sigma)$, and context $X \in |\mathbf{Ctxt}_{\Sigma}^{\mathcal{I}(\mathcal{C})}|$ this interpretation is a function:

$$[\![\_]\!]_{\_}^{\Sigma,M,X} : \mathbf{Frm}_{\Sigma}^{\mathcal{I}(\mathcal{C})}(X) \times \mathbf{Val}_{\Sigma,M}^{\mathcal{I}(\mathcal{C})}(X) \to |Int_{\Sigma}(M)|_{\mathbb{B}}$$

[11] Because, at the level of valuations, metastructures "are" many-sorted algebras.

Let $I(M)$ and $V(M)$ denote the metastructure $Int_\Sigma(M)$ and its *set of values* $V^{Int_\Sigma(M)}$ respectively. We know, that $V(M)$ is a $T(V)$-indexed set, s.t. for every $s \in V$ there is an inclusion $V(M)_{T(s)} \subseteq |I(M)|_{T(s)}$. Let us define:

- for every atomic formula $\phi \in A_\Sigma(X)$, $[\![\phi]\!]_v^{\Sigma,M,X} \mathrel{\widehat{=}} (\phi)^{\overline{v}}_{I(M)}$, where $(\_)^{\overline{v}}_{I(M)}$ denotes the unique extension of the valuation v to the set of terms for the *atomic formulae* signature $\mathbf{Atm}^{\mathfrak{P}}(\Sigma)$,
- $[\![c(\phi_1, \ldots, \phi_n)]\!]_v^{\Sigma,M,X} \mathrel{\widehat{=}} c_{I(M)}([\![\phi_1]\!]_v^{\Sigma,M,X}, \ldots, [\![\phi_n]\!]_v^{\Sigma,M,X})$,
- $[\![qu^\tau\!.\phi]\!]_v^{\Sigma,M,X} \mathrel{\widehat{=}} q_{I(M)}(\{ [\![\phi]\!]_{v[m/u]}^{\Sigma,M,X\cup\{u\}_\tau} \mid m \in V(M)_\tau \})$.

The third clause, defining the semantic interpretation for quantified formulae, perhaps needs some explanation. To obtain the meaning of a quantified formula $qu^\tau\!.\phi$ for the valuation $v \in \mathbf{Val}^{\mathcal{I}(\mathfrak{P})}_{\Sigma,M}(X)$, we first interpret the formula ϕ for all the extensions $v[m/u]$ of v, where m ranges over the appropriate set of values. Then, we apply the generalized operation $q_{I(M)}$ to the subset of $|I(M)|_{\mathbb{B}}$ obtained in this way. Since $\mathfrak{P}$ is logical, $q_{I(M)}$ is total, and thus yields an element of the set $|I(M)|_{\mathbb{B}}$ as a result.

Using the above semantic interpretation for formulae, we can define the *satisfaction relation* for $\mathcal{I}(\mathfrak{P})$. For every signature $\Sigma \in |\mathbf{Sig}^{\mathcal{I}(\mathfrak{P})}|$, model $M \in \mathbf{Mod}^{\mathcal{I}(\mathfrak{P})}(\Sigma)$ and context $X \in |\mathbf{Ctxt}^{\mathcal{I}(\mathfrak{P})}_\Sigma|$ let us define:

$$M[v] \models^{\mathcal{I}(\mathcal{C})}_{\Sigma,X} \phi \quad \text{iff} \quad [\![\phi]\!]_v^{\Sigma,M,X} \in D_{I(M)}.$$

In other words, a formula ϕ is *satisfied* by a valuation v of the context X in the model M, if and only if, the value of its semantic interpretation for v—$[\![\phi]\!]_v^{\Sigma,M,X}$, belongs to the set of *designated elements* of the metastructure $I(M)$.

It can be shown (cf. [14], Th. 23), that the construction sketched above, for every *logical* presentation $\mathfrak{P}$, yields an abstract context institution $\mathcal{I}(\mathfrak{P})$ as a result.

5 Categories of Presentations

In this section we shall introduce a notion of a *presentation morphism* and its refinement, called a *logical presentation morphism*. We shall also describe some structural properties of the corresponding categories of presentations, and discuss how these categories can be used for modular construction of logical systems.

5.1 Presentation Morphisms

An informal idea behind the notion of *presentation morphisms* will be the same as for *abstract context institution morphisms*—we shall think of the source presentation as being an "extension" of the target one.

The reason why morphisms defining such "extensions" are important for us is very simple. The main application of presentation morphisms we have in mind, is a *modular construction* of logical systems (abstract context institutions). Typically, in this process we "enrich" some "atomic" presentations (i.e., presentations having sufficiently simple internal structure), by adding new features and mechanisms, and then “put together” such enrichments.

Definition 9. *Let* $\mathfrak{P} : \mathbf{Sig}^{\mathfrak{P}} \to \mathbf{IntStr}$ *and* $\mathfrak{Q} : \mathbf{Sig}^{\mathfrak{Q}} \to \mathbf{IntStr}$ *be presentations. A morphism from* $\mathfrak{P}$ *to* $\mathfrak{Q}$ *is a pair* $\langle \Phi, \mu \rangle : \mathfrak{P} \to \mathfrak{Q}$*, such that:*

- $\Phi : \mathbf{Sig}^{\mathfrak{P}} \to \mathbf{Sig}^{\mathfrak{Q}}$ *is a functor,*
- $\mu : \Phi ; \mathfrak{Q} \Rightarrow \mathfrak{P}$ *is a natural transformation.*

For every signature $\Sigma \in |\mathbf{Sig}^{\mathfrak{P}}|$, the Σ-component of the natural transformation μ, as a morphism in **IntStr**, consists of:

- a metalanguage signature morphism $\mu_\Sigma^{\mathrm{Lan}} : \mathbf{Lan}^{\mathfrak{Q}}(\Phi(\Sigma)) \to \mathbf{Lan}^{\mathfrak{P}}(\Sigma)$,
- a functor $\mu_\Sigma^{\mathrm{Mod}} : \mathbf{Mod}^{\mathfrak{P}}(\Sigma) \to \mathbf{Mod}^{\mathfrak{Q}}(\Phi(\Sigma))$,
- a natural transformation $\mu_\Sigma^{\mathrm{Int}} : \mu_\Sigma^{\mathrm{Mod}} ; Int_{\Phi(\Sigma)}^{\mathfrak{Q}} \Rightarrow Int_\Sigma^{\mathfrak{P}} ; \mathbf{MStr}(\mu_\Sigma^{\mathrm{Lan}})$.

We shall call a morphism $\langle \Phi, \mu \rangle : \mathfrak{P} \to \mathfrak{Q}$ *logical*, iff for every signature $\Sigma \in |\mathbf{Sig}^{\mathfrak{P}}|$ and every model $M \in \mathbf{Mod}_\Sigma^{\mathfrak{P}}$ the morphism:

$$\mu_\Sigma^{\mathrm{Int}}(M) : Int_{\Phi(\Sigma)}^{\mathfrak{Q}}(\mu_\Sigma^{\mathrm{Mod}}(M)) \to \mathbf{MStr}_{\mu_\Sigma^{\mathrm{Lan}}}(Int_\Sigma^{\mathfrak{P}}(M))$$

satisfies the following conditions:

- the morphism $\mu_\Sigma^{\mathrm{Int}}(M)$ has to both *preserve* (as every metastructure morphism), and *reflect* designated elements.
- $\mu_\Sigma^{\mathrm{Int}}(M)$ has to be *surjective* on values.[12]

Presentations and their morphisms constitute a category, which we shall call the *category of presentations*, and denote by $\mathfrak{Pres}$. The composition of morphisms $\langle \Phi, \mu \rangle : \mathfrak{P} \to \mathfrak{Q}$ and $\langle \Phi', \mu' \rangle : \mathfrak{Q} \to \mathfrak{R}$ in $\mathfrak{Pres}$, is defined as $\langle \Phi ; \Phi', (\Phi * \mu') ; \mu \rangle$.

It is easy to see, that both the identity morphisms and the composition of *logical morphisms* are logical. The subcategory of $\mathfrak{Pres}$, consisting of logical presentations and logical morphisms between them, will be called the *category of logical presentations* and denoted by $\mathfrak{LogPres}$.

Th notion of *logicality* for presentation morphisms, plays the same rôle as it did for presentations. One can show (cf. [14], Th. 24), that the construction of an abstract context institution $\mathcal{I}(\mathfrak{P})$ out of a logical presentation $\mathfrak{P}$, can be extended to logical presentation morphisms, giving for any logical presentation morphism $\langle \Phi, \mu \rangle : \mathfrak{P} \to \mathfrak{Q}$, an abstract context institution morphism $\mathcal{I}(\langle \Phi, \mu \rangle) : \mathcal{I}(\mathfrak{P}) \to \mathcal{I}(\mathfrak{Q})$. It can be shown, that $\mathcal{I}(\_) : \mathfrak{LogPres} \to \mathfrak{AbsConIns}$ is actually a functor.

[12] It has to map the set of values of the metastructure $Int_{\Phi(\Sigma)}^{\mathfrak{Q}}(\mu_\Sigma^{\mathrm{Mod}}(M))$ onto the set of values of the metastructure $\mathbf{MStr}_{\mu_\Sigma^{\mathrm{Lan}}}(Int_\Sigma^{\mathfrak{P}}(M))$.

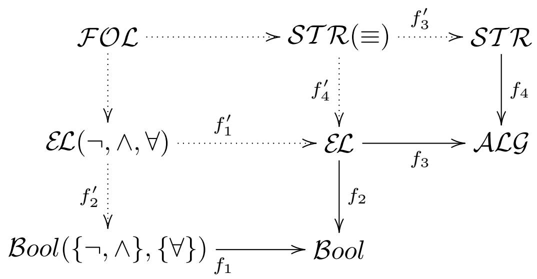
5.2 Limits in Categories of Presentations

As we have mentioned in the Introduction, presentation-like structures called *parchments*, originally invented as a tool for proving the satisfaction condition for ordinary institutions [5], have been later suitably redefined, and proposed as a tool for modular construction of logics [9–11, 13]. In all cases, universal categorical constructions in the appropriate categories of parchments, have been used as a tool. In this section, we would like to show, how this approach can be extended to the case of presentations.

Since *logical presentations* and *logical presentation morphisms* generate abstract context institutions and their morphisms, it would be quite natural to expect, that limits in the category $\mathfrak{LogPres}$ can be used for modular construction of (logical) presentations, and hence—abstract context institutions.

Unfortunately, the category of logical presentations is not complete, so not every diagram in it has a limit. Incompleteness of $\mathfrak{LogPres}$ (cf. [14], Sect. 10.4.3), can be shown along the same lines as incompleteness of the category of *model-theoretic parchments* [11], and *context parchments* [13].

As it turns out, the category $\mathfrak{Pres}$ of presentations is complete (cf. [14], Cor. 14). Since only logical presentations generate abstract context institutions, the question is whether limits in $\mathfrak{Pres}$ can be used as a tool for "putting logics together". Before we answer this question, let us give an example. Let us consider a diagram in $\mathfrak{Pres}$ consisting of four morphisms: f_1, f_2, f_3, f_4:



The morphisms f_1,...,f_4 are the obviously defined "extensions", and $\mathcal{ALG}$ denotes a presentation for the (many-sorted) *equational logic*.

To get a limit in $\mathfrak{Pres}$ of the above diagram, we can first take a limit of the diagram consisting of the morphisms f_1 and f_2. This gives us the presentation $\mathcal{EL}(\neg, \wedge, \forall)$, for an "almost first-order logic with equality ", having algebras as models (i.e., without predicates). Taking a limit of the diagram consisting of f_3 and f_4, we obtain "equational logic for relational structures". Finally, taking the limit of f_1' and f_4' gives us a presentation $\mathcal{FOL}$, for the first-order logic.

All four morphisms from the above diagram (i.e., f_1,...,f_4), are *logical*. Also the result, the presentation $\mathcal{FOL}$, is logical. The example shows, that taking a limit of a logical diagram (i.e., a diagram consisting of logical morphisms between logical presentations), in the category $\mathfrak{Pres}$ *may* lead to a logical presentation as a result.

Unfortunately, it is not always like this. As a very simple example, let us take a pullback of two (logical) extensions of the Boolean presentation $\mathcal{B}ool$: one adding the usual universal quantifier, and the other extending the set of truth values to three elements. In the pullback presentation, the universal quantifier symbol is interpreted by a partial generalized operation—undefined for sets of truth values containing the "third value". Hence, the pullback presentation is not logical.

Four out of five presentations occurring in the above diagram can be seen as "atomic". It is certainly so for the model presentations $\mathcal{ALG}$ and $\mathcal{STR}$, and the Boolean presentation $\mathcal{B}ool$. The presentation $\mathcal{EL}$ for the equational logic may also be seen as "atomic" since there does not seem to exist a meaningful way of constructing it from simpler presentations (equality needs terms, and terms come from algebraic structures). The only "non-atomic" presentation in this diagram is $\mathcal{B}ool(\{\neg, \wedge\}, \{\forall\})$ (it can be constructed from $\mathcal{B}ool$ by "adding" connectives and the quantifier one by one). The pullback of f_1 and f_2 can be seen however as a special case of a more general construction, which we shall call *completion*.

A presentation morphism $\langle \Phi, \mu \rangle : \mathfrak{P} \to \mathfrak{B}$ will be called *Boolean* iff $\mathfrak{B}$ is Boolean, and for every signature $\Sigma \in |\mathbf{Sig}^{\mathfrak{P}}|$, and every Σ-model M, the metastructure morphism:

$$\mu_{\Sigma}^{\mathrm{Int}}(M) : Int_{\Phi(\Sigma)}^{\mathfrak{B}}(\mu_{\Sigma}^{\mathrm{Mod}}(M)) \to \mathbf{MStr}_{\mu_{\Sigma}^{\mathrm{Lan}}}(Int_{\Sigma}^{\mathfrak{P}}(M))$$

is an isomorphism. In other words, a Boolean morphism $\langle \Phi, \mu \rangle : \mathfrak{P} \to \mathfrak{B}$ shows, that the "algebra of truth values" of the presentation $\mathfrak{P}$ is "built over" the algebra of truth values of the Boolean presentation $\mathfrak{B}$. A Boolean morphism $\langle \Phi, \mu \rangle : \mathfrak{P} \to \mathfrak{B}$ will be called a *Boolean extension* if the presentation $\mathfrak{P}$ is Boolean as well.

Let $f : \mathfrak{P} \to \mathfrak{B}$ and $g : \mathfrak{B}' \to \mathfrak{B}$ be a Boolean morphism, and a Boolean extension respectively. We shall call the pullback of f along g a *g-completion of f*. For example, if $g : \mathcal{B}ool(\neg, \wedge) \to \mathcal{B}ool$ is the obvious extension, and $f : \mathfrak{P} \to \mathcal{B}ool$ is a Boolean morphism, then the g-completion of f enriches $\mathfrak{P}$ by adding negation and conjunction. In general, it can be shown that pullbacks of Boolean morphisms along Boolean extensions in $\mathfrak{Pres}$ always exist, and that they are logical.

As it has been already pointed out in [9], and later in [10, 11, 13], one should not expect the "categorical nonsense" to always produce the "expected" result. The main advantage of presentations over other "parchment-like" notions is a clear separation of "logical" and "non-logical" parts. This enables us to define "modularization" techniques, which can always be applied—such as the *completion* construction described above. In general however, the process of modular construction of logics in not (and perhaps will never be) "fully automatic".

6 Concluding Remarks

In the paper, we have described a notion of an *abstract context institution*, generalizing the corresponding notion from [12]. Building upon the work on *parch-*

ments [5, 10, 11] and *context parchments* [13], we have also introduced a notion of *presentation*.

Abstract context institutions enrich institutions [6] by notions of *context*, *substitution* and *valuation*. The valuations are "abstract" objects[13], satisfying some mild technical assumptions. Thanks to their high level of generality, abstract context institutions have numerous examples (see [14]), including systems of *partial*, *modal* and *higher-order* logics. At the same time however, abstract context institutions and their morphisms are *rich enough* to permit some non-trivial constructions. One such construction, of an institution of abstract Hoare logic, has been mentioned in Sect. 3.3 (see also [12, 14]).

Presentations, defined in Sect. 4.3, are structures, which simplify the task of defining abstract context institutions. The notion of presentation is based on the idea of *model-theoretic parchment* [11] and *context parchment* [13]. Many examples of presentations can be found in [14], among them for *total*, *partial*, *modal*, *many-valued* and *higher-order* logics. In Sect. 4.4 we have shown, that for any *logical* presentation $\mathfrak{P}$ we can construct an abstract context institution $\mathcal{I}(\mathfrak{P})$ "generated" by $\mathfrak{P}$. This construction extends to *logical* presentation morphisms.

As it turns out, the category of presentations $\mathfrak{Pres}$, can be used as a framework for modular construction of logics. Presentations share advantages of both model-theoretic parchments and context parchments in this respect.

Thanks to their similarity to model-theoretic parchments, presentations cleanly separate model-theoretic part of a logic from its "syntactic" part. This feature allows for more refined handling of logical syntax in the process of putting presentations together. Furthermore, because the interpretation function takes individual models into account, presentations are capable of describing logics which hardly fit into the standard (context) parchment framework. One example could be a general modal logic, where the space of truth values varies from model to model and depends on a *frame* being a part of the model (cf. [11, 14]).

Similarly to context parchments, presentations allow to "separate" the *algebra of truth values* from the rest of the structure, making it possible to define operations which cannot be defined and performed within the standard (model-theoretic) parchment framework. An interesting example of such a construction—the *closure* operation has been described in Sect. 5.2. The same "separation" has another important consequence wrt. (non-context) parchments. Namely, the structure of a presentation for a given logic is usually much simpler than the structure of a parchment for the same logic (even though, presentations contain an extra information for contexts and substitutions).

We believe, that presentations can be used not only for modular construction of (abstract context) institutions, but also for "putting together" inference systems.

[13] In the case of context institutions [12], valuations were required to be *total* functions from *variables* to *model carriers*.

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