

# Presenting and Combining Inference Systems

– presentations with inference rules –\*

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**Abstract.** The paper discusses the problem of representing and combining inference systems for (abstract) *context institutions*, within the framework of *context presentations* [9]. As it turns out, thanks to the context information present in this setting, the inference rules for quantifier logics can be expressed and manipulated in a simple way, without referring to *binding operators* or *requirements* (cf. [11]).

## 1 Introduction

Many applications of logic in computing science concern composite domains. As a consequence, it is becoming widely accepted that *practically useful* logical formalisms should have a *compositional* nature, possibly reflecting the structure of the problem domain.

Since the development of the notion of *institution* [4], formalizing “an abstract model theory for specification and programming”, the issue of combining logical systems started to attract interest of researchers. In [13] the notion of *parchment* [3], originally invented as a tool for proving the *satisfaction condition* for institutions, has been proposed as a framework for combining logics. The idea has been later “re-discovered”, further refined and explored in a series of papers [5–7, 2].

Institutions and parchment-like structures describe the structural part of a logical system only. In particular, they do not take inference systems into account. The issue of combining inference systems has been investigated mostly in the context of *fibring logics* [10]. More recently, some of the results from this field have been applied to parchment-like structures in [2] (for the case of propositional logics).

In what follows we shall discuss the problem of representing and combining inference systems for (abstract) *context institutions*, within the framework of *context presentations* [9]. As it turns out, thanks to the context information present in this setting, the inference rules for quantifier logics can be expressed and manipulated in a simple way, without referring to *binding operators* or *requirements* (cf. [11]).

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## 2 Context presentations and context institutions

The aim of this section is mostly to recall some basic notions and facts from [9]. Let us start with *metasignatures* and *metastructures* which *presentations* use for describing the syntactic and semantic aspects of a logical system respectively.

### 2.1 Metalanguage framework

**Definition 1.** A metasignature is a six-tuple  $\Sigma = \langle S, \Omega, \Pi, V, C, Q \rangle$ , such that  $\langle S, \Omega, \Pi \rangle$  is a relational signature,  $V \subseteq S$ ,  $C$  is a family of sets indexed by natural numbers, and  $Q$  is a set.

Informally speaking, metasignatures are just relational signatures enriched by symbols of *logical connectives*, (the family  $C$ ), *quantifier symbols* (the set  $Q$ ), and having a distinguished subset of sort names (the set  $V$ ), for which we want to talk about *variables*. A metasignature morphism consist of a relational signature morphism preserving the *variable-sorts* i.e., elements of  $V$ , and two mappings relating the connectives and quantifier symbols (see [9] for details). The metasignatures and their morphisms (with an obvious definition of composition) constitute a category, which we shall denote it by  $\text{MSIG}$ .

For every metasignature  $L = \langle S, \Omega, \Pi, V, C, Q \rangle$ , by  $\text{Syn}(L)$  we shall denote an algebraic signature  $\langle S \uplus \{\star\}, \Omega^{\text{Syn}} \cup \Pi^{\text{Syn}} \cup C^{\text{Syn}} \rangle$  such that:

- $\omega \in \Omega_{w,s}$  iff  $\omega \in \Omega_{T(w),T(s)}^{\text{Syn}}$
- $\pi \in \Pi_w$  iff  $\pi \in \Pi_{T(w),\mathbb{B}}^{\text{Syn}}$
- $c \in C_n$  iff  $c \in C_{\mathbb{B} \dots \mathbb{B},\mathbb{B}}^{\text{Syn}}$

where  $T(s)$  and  $\mathbb{B}$  denote the injection of  $s \in S$  and  $\star$  into the disjoint union  $S \uplus \{\star\}$  respectively.

The above construction in an obvious way extends to a *syntax functor*:  $\text{Syn} : \text{MSIG} \rightarrow \text{ALGSIG}$ . In what follows, we shall also use two “sub-functors” of the functor  $\text{Syn} - \text{Atm} : \text{MSIG} \rightarrow \text{ALGSIG}$  and  $\text{Trm} : \text{MSIG} \rightarrow \text{ALGSIG}$ , such that:

- $\text{Atm}(\langle S, \Omega, \Pi, V, C, Q \rangle) \cong \langle T(S) \cup \{\mathbb{B}\}, \Omega^{\text{Syn}} \cup \Pi^{\text{Syn}} \rangle$ ,
- $\text{Trm}(\langle S, \Omega, \Pi, V, C, Q \rangle) \cong \langle T(S), \langle \Omega_{w \rightarrow t}^{\text{Syn}} \mid w \in T(S)^* \wedge t \in T(S) \rangle \rangle$ ,

The morphisms  $\text{Atm}_\ell$  and  $\text{Trm}_\ell$  are given by the respective components of the morphism  $\text{Syn}_\ell$ . For every metasignature  $L$ , the signatures  $\text{Atm}(L)$  and  $\text{Trm}(L)$  are *sub-signatures* of  $\text{Syn}(L)$ —hence the informal term “sub-functor”.

**Definition 2.** An  $L$ -metastructure  $\mathcal{A}$  consists of:

- a  $\text{Syn}(L)$ -algebra  $A$ ,
- a  $T(V)$ -indexed set  $V_{\mathcal{A}}$ , such that for every  $s \in V$   $(V_{\mathcal{A}})_{T(s)} \subseteq |A|_{T(s)}$ ,
- a set  $D_{\mathcal{A}}$ , such that  $D_{\mathcal{A}} \subseteq |A|_{\mathbb{B}}$ ,
- for every symbol  $q \in Q$ , a partial function  $q_{\mathcal{A}} : \mathcal{P}(|A|_{\mathbb{B}}) \rightarrow |A|_{\mathbb{B}}$ .

The set  $|\mathcal{A}| \hat{=} |A|$  is called the *carrier* of the metastructure  $\mathcal{A}$ , the subset  $V_{\mathcal{A}}$ —its set of *assignable values* (i.e., elements which can be “ranged-over” by variables), and the set  $D_{\mathcal{A}}$ —its set of *designated elements*. The set  $|\mathcal{A}|_{\mathbb{B}}$ , corresponding to the distinguished sort  $\mathbb{B}$  of the “syntactic” signature  $\text{Syn}(L)$ , plays the rôle of the set of *logical values*.

$L$ -metastructures can be viewed as many-sorted algebras enriched by *generalized operations*—the partial functions corresponding to the symbols from  $Q$ . These operations are meant to represent the semantics of *quantifiers*. Metastructure morphisms are just algebraic homomorphisms preserving both the distinguished subsets (of *values* and *designated elements*) and the generalized operations (cf. [9]). The category of  $L$ -metastructures will be denoted by  $\text{MSTR}(L)$ .

## 2.2 Interpretation structures and presentations

Informally speaking, metasignatures can be used for describing the *grammar* of the language of a logical system while metastructures give the semantics of the language. This idea is formalized by notions of an *interpretation structure* and *presentation*.

**Definition 3.** An interpretation structure is a triple  $\langle L, \mathcal{M}, \text{Int} \rangle$ , consisting of:

- a metalanguage signature  $L \in |\text{MSIG}|$
- a class of models  $\mathcal{M} \in |\text{CLASS}|$ ,
- an interpretation function (*functor*)  $\text{Int} : \mathcal{M} \rightarrow \text{MSTR}(L)$ .

Interpretation structures closely resemble *rooms* for *model-theoretic parchments* (cf. [7]). The main difference is that the latter are built over the notion of ordinary (many-sorted) algebra instead of metastructure.

**Definition 4.** A triple  $\langle \ell, m, \text{int} \rangle$  is an interpretation structure morphism from  $\langle L, \mathcal{M}, \text{Int} \rangle$  to  $\langle L', \mathcal{M}', \text{Int}' \rangle$  iff:

- $\ell : L \rightarrow L'$  is a metasignature morphism,
- $m : \mathcal{M}' \rightarrow \mathcal{M}$  is a function,
- $\text{int} : m ; \text{Int} \Rightarrow \text{Int}' ; \text{MSTR}_{\ell}$  is a natural transformation.

Since  $\mathcal{M}'$  is a discrete category (a class) the natural transformation  $\text{int}$  is simply an  $\mathcal{M}'$ -indexed family of metastructure morphisms, such that for every  $M' \in \mathcal{M}'$ :

$$\text{int}_{M'} : \text{Int}(m(M')) \rightarrow \text{MSTR}_{\ell}(\text{Int}'(M')).$$

The composition of  $\langle \ell, m, \text{int} \rangle : IS_1 \rightarrow IS_2$  and  $\langle \ell', m', \text{int}' \rangle : IS_2 \rightarrow IS_3$  is defined by:

$$\langle \ell, m, \text{int} \rangle ; \langle \ell', m', \text{int}' \rangle \hat{=} \langle \ell ; \ell', m' ; m, (m' * \text{int}) ; (\text{int}' * \text{MSTR}_{\ell}) \rangle.$$

Interpretation structures and their morphisms constitute a category which we shall denote by  $\text{INTSTR}$ .

**Definition 5.** A context presentation is an arbitrary functor into the category of interpretation structures:

$$\mathcal{P} : \text{SIG}^{\mathcal{P}} \rightarrow \text{INTSTR}.$$

We shall call  $\text{SIG}^{\mathcal{P}}$  the *category of signatures* of the presentation  $\mathcal{P}$ .

Let  $\mathcal{P} : \text{SIG}^{\mathcal{P}} \rightarrow \text{INTSTR}$  be a context presentation. To simplify notation, in the sequel we shall omit the superscript from the name of the category of signatures for  $\mathcal{P}$ .

For any signature  $\Sigma \in |\text{SIG}|$  the presentation  $\mathcal{P}$  assigns an interpretation structure  $\mathcal{P}(\Sigma)$ :

$$\Sigma \mapsto \mathcal{P}(\Sigma) = \langle L_{\Sigma}, \mathcal{M}_{\Sigma}, \text{Int}_{\Sigma} : \mathcal{M}_{\Sigma} \rightarrow \text{MSTR}(L_{\Sigma}) \rangle.$$

By “projecting”  $\mathcal{P}$  to the first and second component respectively we obtain the *metalanguage functor*  $\text{Lan}^{\mathcal{P}} : \text{SIG} \rightarrow \text{MSIG}$  and the *model functor*  $\text{Mod}^{\mathcal{P}} : \text{SIG}^{op} \rightarrow \text{CLASS}$  for  $\mathcal{P}$ . Relationships between these two functors can be depicted as follows:

$$\begin{array}{ccc} \Sigma \mapsto \text{Lan}_{\Sigma}^{\mathcal{P}} & & \text{Mod}_{\Sigma}^{\mathcal{P}} \xrightarrow{\text{Int}_{\Sigma}} \text{MSTR}(\text{Lan}_{\Sigma}^{\mathcal{P}}) \\ \sigma \downarrow & \text{Lan}_{\sigma}^{\mathcal{P}} \downarrow & \text{Mod}_{\sigma} \uparrow \searrow \text{int}_{\sigma} \\ \Sigma' \mapsto \text{Lan}_{\Sigma'}^{\mathcal{P}} & & \text{Mod}_{\Sigma'}^{\mathcal{P}} \xrightarrow{\text{Int}_{\Sigma'}} \text{MSTR}(\text{Lan}_{\Sigma'}^{\mathcal{P}}) \end{array}$$

**Definition 6.** Let  $\mathcal{P}_1 : \text{SIG}^{\mathcal{P}_1} \rightarrow \text{INTSTR}$  and  $\mathcal{P}_2 : \text{SIG}^{\mathcal{P}_2} \rightarrow \text{INTSTR}$  be presentations. A morphism from  $\mathcal{P}_1$  to  $\mathcal{P}_2$  is a pair  $\langle \Phi, \mu \rangle : \mathcal{P}_1 \rightarrow \mathcal{P}_2$  such that:

- $\Phi : \text{SIG}^{\mathcal{P}_1} \rightarrow \text{SIG}^{\mathcal{P}_2}$  is a functor,
- $\mu : \Phi ; \mathcal{P}_2 \Rightarrow \mathcal{P}_1$  is a natural transformation.

Presentations and presentation morphisms constitute a category which we shall denote by  $\text{PRES}$ .

We shall now give two simple examples and describe a presentation  $\mathcal{EQL}$  for the *many-sorted equational logic* and presentation  $\mathcal{FOL}$  for the *many-sorted first-order logic* (without equality). Other examples of presentations, as well as examples of their morphisms, can be found in [8, 9].

*Example 1 (Many-sorted equational logic).* The category of signatures for  $\mathcal{EQL}$  is the category of algebraic signatures  $\text{ALGSIG}$ . The metalanguage functor  $\text{Lan}^{\mathcal{EQL}} : \text{ALGSIG} \rightarrow \text{MSIG}$  is defined as follows:

- $\text{Lan}^{\mathcal{EQL}}(\langle S, \Omega \rangle) \hat{=} \langle S, \Omega, \Pi, S, C, \emptyset \rangle$ , where  $\Pi_{ss} \hat{=} \{\equiv\}$  for  $s \in S$ , and all other elements of  $\Pi$ , and all  $C_n$  for  $n \geq 0$  are empty.
- for every  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  in  $\text{ALGSIG}$ , the morphism  $\text{Lan}_{\sigma}^{\mathcal{EQL}} : \text{Lan}^{\mathcal{EQL}}(\Sigma_1) \rightarrow \text{Lan}^{\mathcal{EQL}}(\Sigma_2)$  is defined as  $\sigma$  for the symbols coming from  $\Sigma$ , and as the identity for the *equations* from  $\Pi_{ss}$ , for  $s \in S$ .

The model functor  $\mathbf{Mod}^{\mathcal{EQL}}$  for every signature  $\Sigma$  in  $\mathbf{ALGSIG}$  returns the class of all  $\Sigma$ -algebras  $\mathbf{ALG}(\Sigma)$ . For every algebraic signature morphism  $\sigma$ , the function  $\mathbf{Mod}_\sigma^{\mathcal{EQL}}$  is the usual algebraic  $\sigma$ -reduct operation.

Let  $\Sigma = \langle S, \Omega \rangle$  be an algebraic signature. For every  $\Sigma$ -model (algebra)  $M$ , we shall now define the metastructure  $\mathit{Int}_\Sigma(M)$ . Let  $A_M$  be a  $\mathbf{Syn}^{\mathcal{EQL}}(\Sigma)$ -algebra such that:

- for every sort  $s \in S$ , let  $|A_M|_{T(s)} \hat{=} |M|_s$ ,
- $|A_M|_{\mathbb{B}} \hat{=} \{\mathbf{tt}, \mathbf{ff}\}$ ,
- for every  $\omega : s_1 \dots s_n \rightarrow s_0$ ,  $\omega_{A_M} \hat{=} \omega_M$ ,
- for every  $s$  in  $S$  and  $a_1, a_2$  in  $|A_M|_{T(s)}$ ,  $\equiv_{A_M}(a_1, a_2)$  equals  $\mathbf{tt}$  if and only if  $a_1$  and  $a_2$  are identical.

For every  $s \in S$ , as the *set of values*  $|V^{\mathit{Int}_\Sigma(M)}|_{T(s)}$  for the sort  $T(s)$ , let us take the whole<sup>1</sup> set  $|A_M|_{T(s)}$ , and as the set of *designated elements*  $D^{\mathit{Int}_\Sigma(M)}$ —the singleton set  $\{\mathbf{tt}\}$ .

For every algebraic signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  and every algebra  $A \in \mathbf{Mod}_{\Sigma_2}^{\mathcal{EQL}}$ , let  $\mathit{int}_A^\sigma$  be the identity morphism.

*Example 2 (Many-sorted first-order logic).* The category of signatures for  $\mathcal{FOL}$  is the category of *relational signatures*  $\mathbf{RELSIG}$ , whose objects are triples  $\langle S, \Omega, \Pi \rangle$ , where  $S$  is the set of *sort names*,  $\Omega$  is an  $S^* \times S$ -indexed set of *operation symbols* and  $\Pi$  is an  $S^*$ -indexed set of *relational symbols*. Morphisms in  $\mathbf{RELSIG}$  are defined in the usual way. The metalanguage functor  $\mathbf{Lan}^{\mathcal{FOL}} : \mathbf{RELSIG} \rightarrow \mathbf{MSIG}$  is defined as follows:

- $\mathbf{Lan}^{\mathcal{FOL}}(\langle S, \Omega, \Pi \rangle) \hat{=} \langle S, \Omega, \Pi, S, C, \{\forall\} \rangle$ , where  $C_1 \hat{=} \{\neg\}$ ,  $C_2 \hat{=} \{\wedge\}$ , and the sets:  $C_0$  and  $C_n$  for  $n \geq 3$  are empty,
- for every relational signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$ , the morphism  $\mathbf{Lan}_\sigma^{\mathcal{FOL}} : \mathbf{Lan}^{\mathcal{FOL}}(\Sigma_1) \rightarrow \mathbf{Lan}^{\mathcal{FOL}}(\Sigma_2)$  is given by  $\sigma$ , for symbols coming from  $\Sigma$ , and is defined as the identity for *connectives* and the *quantifier*, i.e. for the symbols:  $\neg, \wedge$  and  $\forall$ .

The functor  $\mathbf{Mod}^{\mathcal{FOL}} : \mathbf{RELSIG}^{op} \rightarrow \mathbf{CLASS}$ , for every relational signature  $\Sigma$ , returns the class of all *relational  $\Sigma$ -structures* (defined in a standard way). For every  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  in  $\mathbf{RELSIG}$ , the function  $\mathbf{Mod}_\sigma^{\mathcal{FOL}}$  is the corresponding  $\sigma$ -reduct operation.

Let us now define the interpretation function for  $\Sigma = \langle S, \Omega, \Pi \rangle$ . Let  $M$  be an arbitrary relational  $\Sigma$ -structure. We have to define the metastructure  $\mathit{Int}_\Sigma(M)$ . Let  $A_M$  be a  $\mathbf{Syn}^{\mathcal{FOL}}(\Sigma)$ -algebra, such that:

- the *carrier*  $|A_M|$  and the interpretation of operation symbols from  $\Omega$  are defined as in the case of  $\mathcal{EQL}$ ,
- for every relational symbol  $\pi : s_1 \dots s_n$  and  $a_i$  in  $|A_M|_{T(s_i)}$ ,  $i = 1 \dots n$  the value of  $\pi_{A_M}(a_1, \dots, a_n)$  equals  $\mathbf{tt}$  if and only if the tuple  $\langle a_1, \dots, a_n \rangle$  belongs to  $\pi_M$ ,

<sup>1</sup> In general, the set of values could be a proper subset of the carrier, e.g. as in the case of *partial* first-order logic (cf. [9])

–  $\neg_{A_M}$  and  $\wedge_{A_M}$  are the usual negation and conjunction.

The *set of values* and the *set of designated elements* are defined in the same way as for  $\mathcal{EQL}$ . The generalized operation  $\forall_{Int_{\Sigma}(M)}$ , interpreting the  $\forall$  quantifier is defined as follows:

$$\forall_{Int_{\Sigma}(M)}(B) \hat{=} \begin{cases} \mathbf{ff} & \text{if } \mathbf{ff} \in B \\ \mathbf{tt} & \text{otherwise.} \end{cases}$$

where  $B \subseteq |A_M|_{\mathbb{B}}$  is an arbitrary subset.

For every signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  in  $\text{RELSIG}$ , and every model  $M_2 \in \text{Mod}_{\Sigma_2}^{\text{FOL}}$ , the morphis  $int_{M_2}^{\sigma}$  is the identity.

### 2.3 Towards syntax—metaformulae

As we have mentioned, we want to treat metasignatures as *grammars* for the languages of terms and formulae. This idea is made precise through the construction of a *metaformula functor*

$$\mathbf{MFrm} : \text{MSig} \rightarrow \text{sDGM}(\text{SET})$$

where  $\text{sDGM}(\text{SET})$  is the category of *small diagrams* in  $\text{SET}^2$ . Below, we shall briefly sketch this construction, since (a slight modification of) it will also be used in the next section for defining *schematic metaformulea*.

Let  $\mathcal{V}$  be an arbitrary (denumerable) vocabulary of *variable symbols* with a fixed choice function *choice* :  $(\mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}) \rightarrow \mathcal{V}$ , i.e. a function such that *choice*( $V$ )  $\in V$  for all nonempty  $V \subseteq \mathcal{V}$ .

For every metasignature  $L = \langle S, \Omega, \Pi, V, C, Q \rangle$  let  $\text{MCTXT}_L$  be the full subcategory of the category of substitutions  $\mathcal{T}_{\text{Trm}(L)}$ , whose objects are  $T(S)$ -sorted sets  $X$  of elements of  $\mathcal{V}$  such that  $X_{T(s)} = \emptyset$  for  $s \notin V$ . This category will be called the category of *L-metacontexts*.

For any  $L$ -metacontext  $X$ , using (almost) the usual first-order syntax approach, we define the set of metaformulae  $\mathbf{MFrm}_L(X)$ . The construction proceeds by induction over the set of all  $L$ -metacontexts. Let  $\{\mathbf{MFrm}_L(X) \mid X \in |\text{MCTXT}_L|\}$  be the smallest family of sets satisfying the following conditions:

$$|T_{\text{Atm}(L)}(X)|_{\mathbb{B}} \subseteq \mathbf{MFrm}_L(X)$$

$$\frac{c \in C_n \quad \varphi_1 \dots \varphi_n \in \mathbf{MFrm}_L(X)}{c(\varphi_1, \dots, \varphi_n) \in \mathbf{MFrm}_L(X)}$$

$$\frac{\varphi \in \mathbf{MFrm}_L(X \cup \{v\}_s) \quad v = \text{fresh}(X) \quad q \in Q \quad s \in V}{q v^s. \varphi \in \mathbf{MFrm}_L(X)}$$

<sup>2</sup> The objects in  $\text{sDGM}(\text{SET})$  are pairs  $\langle A, F \rangle$ , where  $A$  is a *small* category and  $F : A \rightarrow \text{SET}$  is a functor. A morphism from  $\langle A, F \rangle$  to  $\langle B, G \rangle$  is a pair  $\langle H, \alpha \rangle$ , such that  $H : A \rightarrow B$  is a functor and  $\alpha : F \Rightarrow H; G$  is a natural transformation. Composition of morphisms  $\langle H_1, \alpha \rangle : \langle A_1, F_1 \rangle \rightarrow \langle A'_1, F'_1 \rangle$  and  $\langle H_2, \beta \rangle : \langle A_2, F_2 \rangle \rightarrow \langle A'_2, F'_2 \rangle$  is defined as  $\langle H_1; H_2, \alpha; (H_1 * \beta) \rangle$ .

where  $fresh(X) \approx choice(\mathcal{V} \setminus |X|)$ . The first condition expresses the fact that every *atomic metaformula* is a metaformula. The remaining two requirements say that the set  $\mathbf{MFrm}_L(X)$  is closed under applications of *connectives* and *quantifiers*.

The requirement that  $v = fresh(X)$  guarantees that the resulting metaformulae are *normalized* wrt. a suitably defined notion of syntactic substitution (in the sense of [12, 8]). Hence, we can avoid the complexity of working with equivalence classes (wrt.  $\alpha$ -conversion) of metaformulae. The syntax translation along metasignature morphisms is defined in a standard way. It is not difficult to show that the construction indeed defines a functor from  $\mathbf{MSIG}$  to  $\mathbf{SDGM}(\mathbf{SET})$ .

By the *base* of an arbitrary functor  $F : \mathbf{A} \rightarrow \mathbf{SDGM}(\mathbf{SET})$  we shall mean a functor  $\mathbf{base}(F) : \mathbf{A} \rightarrow \mathbf{SCAT}$ , such that  $\mathbf{base}(F)(A) \hat{=} C_A$  and  $\mathbf{base}(a : A_1 \rightarrow A_2) \hat{=} C_a$ , where  $F(A) = C_A : C_A \rightarrow \mathbf{SET}$  and  $F(a) = \langle C_a : C_{A_1} \rightarrow C_{A_2}, \alpha \rangle$ . For example, the base for the metaformula functor  $\mathbf{MFrm}$  is the *metacontext functor*  $\mathbf{MCtxt} : \mathbf{MSIG} \rightarrow \mathbf{SCAT}$ , which to every metasignature  $L$  assigns the category of  $L$ -metacontexts  $\mathbf{MCtxt}_L$  and for every metasignature morphism gives the appropriate metacontext translation.

## 2.4 From presentation to context institution

In [8, 9] universal constructions in the category  $\mathbf{PRES}$  of presentations have been proposed as a framework for systematic construction of logical systems. Among the objects of  $\mathbf{PRES}$  the “really interesting ones” are those for which we can directly construct a corresponding *context institution*.

**Definition 7.** A context institution *consists of*:

- a category  $\mathbf{SIG}$  of signatures,
- a formula functor  $\mathbf{Frm} : \mathbf{SIG} \rightarrow \mathbf{SDGM}(\mathbf{SET})$
- a model functor  $\mathbf{Mod} : \mathbf{SIG}^{op} \rightarrow \mathbf{CLASS}$ ,
- a valuation functor  $\mathbf{Val} : \mathbf{ELTS}(\mathbf{Mod}) \rightarrow \mathbf{SDGM}(\mathbf{SET})$ ,

such that  $\mathbf{base}(\mathbf{Val}); (-)^{op} = \mathbf{base}(\pi; \mathbf{Frm})$ , where  $\mathbf{ELTS}(\mathbf{Mod})$  denotes the category of elements<sup>3</sup> for  $\mathbf{Mod}$  and  $\pi : \mathbf{ELTS}(\mathbf{Mod}) \rightarrow \mathbf{SIG}$  is the obvious projection functor, plus for every: signature  $\Sigma$ , model  $M$  in  $\mathbf{Mod}_\Sigma$ , and context  $\Gamma \in |\mathbf{Ctx}_\Sigma|$  (where  $\mathbf{Ctx} \hat{=} \mathbf{base}(\mathbf{Frm})$ ),

- a binary satisfaction relation:

$$M[-] \models_{\Sigma, \Gamma} - \subseteq \mathbf{Val}_{\Sigma, M}(\Gamma) \times \mathbf{Frm}_\Sigma(\Gamma)$$

such that suitably defined Satisfaction and Substitution conditions hold (see [9] for details).

<sup>3</sup> The objects in  $\mathbf{ELTS}(\mathbf{Mod})$  are pairs  $\langle \Sigma, M \rangle$  s.t.  $\Sigma \in |\mathbf{SIG}|$  and  $M \in |\mathbf{Mod}_\Sigma|$ . A morphism from  $\langle \Sigma_1, M_1 \rangle$  to  $\langle \Sigma_2, M_2 \rangle$  is a signature morphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  s.t.  $\mathbf{Mod}_\sigma(M_2) = M_1$  (for more on categories of elements see [1]).

Due to the lack of space we shall not give any examples of context institutions here. Interested Reader may consult [9, 8] for more information, motivation as well as the definition of *context institution morphism*. In what follows the category of context institutions will be denoted by  $\text{CONINS}$ .

The condition which enables us to construct a context institution  $\mathcal{I}(\mathcal{P})$  out of a context presentation  $\mathcal{P}$  is called *logicality*. In what follows  $\mathcal{I}(\mathcal{P})$  will be called the context institution *generated by  $\mathcal{P}$* .

**Definition 8.** A presentation  $\mathcal{P} : \text{SIG} \rightarrow \text{INTSTR}$  is called *logical iff*:

- for every signature  $\Sigma$  in  $\text{SIG}$ , and every model  $M$  in  $\text{Mod}_\Sigma$ , all the generalized operations in the metastructure  $\text{Int}_\Sigma(M)$  are total,
- for every signature morphism  $\sigma : \Sigma \rightarrow \Sigma'$  and every model  $M'$  in  $\text{Mod}_{\Sigma'}$  the morphism  $\text{int}_{M'}^\sigma : \text{Int}_\Sigma(\text{Mod}_\sigma(M')) \rightarrow \text{MSTR}_{\text{Lan}_\sigma}(\text{Int}_{\Sigma'}(M'))$ :
  - reflects the designated elements,
  - is surjective on the set of assignable values.

The notion of logicality extends to presentation morphisms yielding a category of *logical presentations* denoted by  $\text{LOGPRES}$ .

For an arbitrary logical presentation  $\mathcal{P}$  the context institution  $\mathcal{I}(\mathcal{P})$  generated by  $\mathcal{P}$ , has the same category of signatures as  $\mathcal{P}$ . Also the model functor for  $\mathcal{I}(\mathcal{P})$  is the model functor for  $\mathcal{P}$  (see Sect. 2.2). The *formula functor* is defined as the composition  $\text{Lan}^\mathcal{P}; \text{MFrm}$ . Valuations are (suitably indexed) functions between contexts and *carriers* of the metastructures interpreting models (given by  $\text{Int}$ ). For every signature  $\Sigma$ , every  $\Sigma$ -model  $M$  and every  $\Sigma$ -context  $X$  the satisfaction relation is given by:

$$M[v] \models_{\Sigma, X}^{\mathcal{I}(\mathcal{P})} \phi \quad \text{iff} \quad \llbracket \phi \rrbracket_v \in D_{\text{Int}_\Sigma(M)}.$$

where  $\llbracket \_ \rrbracket_v$  denotes a *semantic interpretation function* for formulae.

In other words, a formula  $\phi$  is *satisfied* by a valuation  $v$  of the context  $X$  in the model  $M$ , if and only if, the value of its semantic interpretation corresponding to  $v$  belongs to the set of *designated elements* of the metastructure  $\text{Int}_\Sigma(M)$  (see [9, 8] for details). The construction of  $\mathcal{I}(\mathcal{P})$  extends to a functor  $\mathcal{I}(\_ ) : \text{LOGPRES} \rightarrow \text{CONINS}$  (cf. [8], Theorem 23 and Proposition 24).

In an obvious way both presentations described above—*EQL* and *FOL*—are logical. Many more examples of logical presentations can be found in [8, 9].

### 3 Schematic metaformulae

To proceed towards our goal of introducing *inference rules* for context presentations we need to define expressions which, when appropriately “instantiated”, would denote formulae of the object logic. We shall call such expressions *schematic metaformulae*. They constitute a functor

$$\text{SMFrm} : \text{MSIG} \rightarrow \text{SDGM}(\text{SET}) \tag{1}$$

which we define below. In what follows we shall frequently use the term “s-metaformula” instead of “schematic metaformula”.

In addition to the “ordinary” variables (elements of  $\mathcal{V}$ ) s-metaformulae can contain variables denoting formulae of the logic in question. Those *formula variables* have to constitute a part of the extended notion of *context* for s-metaformulae. Defining *schematic metacontexts* we have to be careful to avoid “incompatibilities” between different occurrences of the same formula variable in a given s-metaformula. To clarify it let us consider the expression

$$qx^\tau.\phi \Rightarrow qx^\sigma.\phi \quad (2)$$

containing two occurrences of a formula variable  $\phi$ .

Assume that we want the whole expression to be interpreted in the set of metaformulae over a given metacontext  $X$ . Then, because of the way the family of sets  $\mathbf{MFrm}_L(X)$  has been defined, the first occurrence of  $\phi$  should be evaluated within the set of metaformulae over  $X \cup \{x\}_\tau$ . At the same time, for the same reason the second occurrence has to be evaluated in the set of metaformulae over  $X \cup \{x\}_\sigma$ <sup>4</sup>. In general these two sets would be different. Since in a given instance of the quasi s-metaformula (2) we want both occurrences of  $\phi$  to denote the same metaformula we shall put the information about the *binding context* of a formula variable into the schematic metacontext.

Let  $\mathcal{FV}$  be the dictionary (set) of *formula variables*. We shall assume that  $\mathcal{V}$  and  $\mathcal{FV}$  are disjoint. For any metasignature  $L = \langle S, \Omega, \Pi, V, C, Q \rangle$  by  $\mathbf{SMCTXT}_L$  we shall denote a category whose objects are triples  $\langle \bar{s}, X, \Phi \rangle$  such that  $\bar{s} \in V^*$ ,  $X$  is an  $L$ -metacontext and  $\Phi$  is an  $V^*$ -sorted set of elements from  $\mathcal{FV}$ . We shall also say, that a formula variable  $\phi \in \Phi_{s_1 \dots s_n}$  has a *binding context*  $s_1 \dots s_n$ . A morphism in  $\mathbf{SMCTXT}_L$  from  $\langle \bar{s}, X, \Phi \rangle$  to  $\langle \bar{s}, Y, \Psi \rangle$ , where  $\Phi \subseteq \Psi$  is an arbitrary metacontext morphism  $t : X \rightarrow Y$ .

In what follows, for any metacontext  $X$  and any  $s \in V$  by  $s(X)$  we shall denote the metacontext  $X \cup \{fresh(X)\}_s$ . This notation extends to any sequence  $\bar{s} = s_1 \dots s_n$  by putting  $\bar{s}(X) = s_1(\dots s_n(X) \dots)$ .

Let  $\{\mathbf{SMF}_L\langle \bar{s}, X, \Phi \rangle \mid \langle \bar{s}, X, \Phi \rangle \in |\mathbf{SMCTXT}_L|\}$  denotes the least family of sets such that  $\mathbf{SMF}_L\langle \bar{s}, X, \Phi \rangle \subseteq \mathbf{SMF}_L\langle \bar{s}, X, \Psi \rangle$  for  $\Phi \subseteq \Psi$  and satisfying the following additional closure conditions:

$$\frac{\bar{s} \in V^* \quad \varphi \in |T_{\text{Atm}(L)}(\bar{s}(X))|_{\mathbb{B}}}{\varphi \in \mathbf{SMF}_L\langle \bar{s}, X, \emptyset \rangle} \quad \frac{\phi \in \mathcal{FV} \quad \bar{s} \in V^*}{\phi \in \mathbf{SMF}_L\langle \bar{s}, X, \{\phi\}_{\bar{s}} \rangle}$$

$$\frac{c \in C_n \quad \varphi_1 \dots \varphi_n \in \mathbf{SMF}_L\langle \bar{s}, X, \Phi \rangle}{c(\varphi_1, \dots, \varphi_n) \in \mathbf{SMF}_L\langle \bar{s}, X, \Phi \rangle}$$

$$\frac{\varphi \in \mathbf{SMF}_L\langle s::\bar{s}, X, \Phi \rangle \quad v = fresh(\bar{s}(X)) \quad q \in Q \quad s \in V}{qv^s.\varphi \in \mathbf{SMF}_L\langle \bar{s}, X, \Phi \rangle}$$

<sup>4</sup> More formally, instead of the variable  $x$  we should have used  $fresh(X)$  (with appropriate sort “decoration”).

$$\frac{\varphi \in \mathbf{SMF}_L \langle \bar{s}, X, \Phi \rangle \quad s \in V}{[\varphi]_s \in \mathbf{SMF}_L \langle s::\bar{s}, X, \Phi \rangle}$$

The last clause represents an “inclusion”  $[-]_s$  between the sets  $\mathbf{SMF}_L \langle \bar{s}, X, \Phi \rangle$  and  $\mathbf{SMF}_L \langle s::\bar{s}, X, \Phi \rangle$ .<sup>5</sup>

As in the case of metaformulae, because of the “freshness” requirement for the variable  $v$  above, each of the sets  $\mathbf{SMF}_L \langle \bar{s}, X, \Phi \rangle$  contains only expressions which are *normalized* wrt. syntactic substitution of terms for variables, i.e. expressions invariant under the identity substitution.

For any metasignature  $L$  the *schematic metaformula functor*  $\mathbf{SMFrm}_L$  to every schematic metacontext  $\langle \bar{s}, X, \Phi \rangle$  assigns the set  $\mathbf{SMF}_L \langle \bar{s}, X, \Phi \rangle$ . For any s-metacontext morphism  $t : \langle \bar{s}, X, \Phi \rangle \rightarrow \langle \bar{s}, Y, \Psi \rangle$ , its image under  $\mathbf{SMFrm}_L$  is a suitably defined substitution operation. Due to the lack of space we shall not give its almost obvious definition here. For a metasignature morphism  $\ell : L_1 \rightarrow L_2$  its image under  $\mathbf{SMFrm}$  is a family of functions translating for every s-metacontext  $\langle \bar{s}, X, \Phi \rangle$  s-metaformulae from  $\mathbf{SMF}_{L_1} \langle \bar{s}, X, \Phi \rangle$  to the set  $\mathbf{SMF}_{L_2} \langle \ell^*(\bar{s}), \text{MCtxt}_\ell(X), \text{SSet}_{\ell^*}(\Phi) \rangle$ .<sup>6</sup>

Note, that there is a “natural inclusion”  $\alpha : \mathbf{Mfrm} \Rightarrow \mathbf{SMfrm}$ . For every metasignature  $L$  the functor  $\alpha_L^{ctx} : \text{MCtxt}_L \rightarrow \text{SMCtxt}_L$  sends each metacontext  $X$  to the s-metacontext  $\langle \epsilon, X, \emptyset \rangle$  and every morphism  $f : X \rightarrow Y$  to itself. For any metacontext  $X$  the function  $\alpha_L^{frm}(X) : \mathbf{Mfrm}_L(X) \rightarrow \mathbf{SMfrm}_L \langle \epsilon, X, \emptyset \rangle$  is the identity.

## 4 Inference rules and extended presentations

Let  $L$  be an arbitrary metasignature. Taking schematic metaformulae as the basis, we shall now define the notion of an *L-inference-rule*.

**Definition 9.** *An L-inference-rule is a triple  $r = \langle \langle \bar{s}, X, \Phi \rangle, \text{prem}(r), \text{conc}(r) \rangle$  such that:*

- $\langle \bar{s}, X, \Phi \rangle$  is a schematic metacontext,
- $\text{prem}(r) \in \mathcal{P}(\mathbf{SMfrm}_L \langle \bar{s}, X, \Phi \rangle)$ ,
- $\text{conc}(r) \in \mathbf{SMfrm}_L \langle \bar{s}, X, \Phi \rangle$

Using translations of s-metacontexts and s-metaformulae provided by the functor  $\mathbf{SMfrm}$  we can easily see that the inference rules actually define a functor

$$\text{Rules} : \text{MSIG} \rightarrow \text{SET}$$

which for every metasignature  $L$  assigns the set of all  $L$ -rules.

<sup>5</sup> Technically speaking  $[-]_s$  is an obvious extension (by taking *formula variables* into account) of the operation of performing an “inclusion substitution”  $\iota : X \rightarrow s(X)$ . In particular,  $[-]_s$  “re-normalizes” its argument wrt.  $s(X)$ .

<sup>6</sup> By  $\ell^*$  we denote the obvious extension of the *sort-component* of  $\ell$  to sequences.  $\text{SSet}$  is the functor “creating” categories of *sorted sets* (cf. [9]).

**Definition 10.** An extended interpretation structure is a pair  $\langle IS, R \rangle$  consisting of:

- an interpretation structure  $IS = \langle L, \mathcal{M}, Int \rangle$ ,
- a set of  $L$ -rules  $R \subseteq \text{Rules}(L)$ .

A morphism from  $\langle IS_1, R_1 \rangle$  to  $\langle IS_2, R_2 \rangle$  is an arbitrary morphism  $\langle \ell, m, int \rangle : IS_1 \rightarrow IS_2$  in  $\text{INTSTR}$  such that  $\text{Rules}_\ell(R_1) \subseteq R_2$ .

The category of *extended interpretation structures* and their morphisms will be denoted by  $\text{EINTSTR}$ .

**Proposition 1.** The category  $\text{EINTSTR}$  is cocomplete.

*Proof.* The category  $\text{INTSTR}$  is cocomplete ([8], Theorem 13). Similarly cocomplete is the category of metasignatures ([8], Proposition 18). Using these two facts it is easy to verify that the colimits in  $\text{EINTSTR}$  are computed via colimits in  $\text{INTSTR}$  and a suitable category of “signed” sets of rules.

**Definition 11.** An extended context presentation is an arbitrary functor into the category of extended interpretation structures:

$$\mathcal{P} : \text{SIG}^{\mathcal{P}} \rightarrow \text{EINTSTR}.$$

Defining the morphisms similarly as in the case of (ordinary) presentations we obtain a category of *extended presentations*  $\text{EPRES}$ .

**Corollary 1.** The category  $\text{EPRES}$  of extended presentations is complete.

*Proof.* Completeness is a consequence of the fact that  $\text{EPRES}$  is a category of functors “into” a cocomplete category.

*Example 3 (Many-sorted equational logic contd.).* Let us take the presentation  $\mathcal{EQL}$  for many-sorted equational logic, defined in Example 1. We shall turn it into an extended presentation  $\mathcal{EQL} : \text{ALGSIG} \rightarrow \text{EINTSTR}$  by augmenting for every algebraic signature  $\Sigma = \langle S, \Omega \rangle$  the original interpretation structure  $\mathcal{EQL}(\Sigma)$  by the set of rules  $R_\Sigma$ , consisting of the *reflexivity*, *symmetry*, *transitivity* and *congruence* rules. For example for all  $s \in S$  the transitivity rule looks as follows:

$$\frac{x^s \equiv y^s \quad y^s \equiv z^s}{x^s \equiv z^s} \langle \epsilon, \{x, y, z\}_s, \emptyset \rangle$$

Please note, that there is no *substitution rule* among the rules in  $R_\Sigma$ . It may seem strange at first, but as we shall learn in the next section, there is a good reason for this “omission”.

*Example 4 (Many-sorted first-order logic contd.).* To obtain an extended presentation for the first-order logic (without equality) it is enough to augment each interpretation structure for the presentation  $\mathcal{FOL}$  by inference rules corresponding to the axiom schemata of the classical propositional calculus, the rule of *modus ponens* and the following rules:

$$\frac{}{[\forall x^s.\phi]_s \Rightarrow \phi} \langle s, \emptyset, \{\phi\}_s \rangle \quad \frac{[\phi]_s \Rightarrow \psi}{[\phi]_s \Rightarrow [\forall x^s.\phi]_s} \langle s, \emptyset, \{\{\phi\}_\epsilon, \{\psi\}_s\} \rangle$$

for all  $s \in S$ .

#### 4.1 From extended presentations to entailment systems

The notion of *logicality* introduced for ordinary presentations can without any change be used for the extended case—we do not require anything extra about the inference rules of *extended logical presentations*. We shall denote the resulting category by  $\text{ELOGPRES}$ .

Let  $\mathcal{P} : \text{SIG} \rightarrow \text{EINTSTR}$  be a logical extended presentation. In this section we shall define the *entailment system* generated by the rules of  $\mathcal{P}$  and discuss its semantic interpretation within the context institution  $\mathcal{I}(\mathcal{P})$ . Let us start with the notion of a *valuation* of a *schematic metacontext*.

**Definition 12.** A valuation of a schematic metacontext  $\langle \bar{s}, X, \Phi \rangle$  in an  $L$ -metacontext  $Y$  is an arbitrary pair  $\langle t, \rho \rangle$  such that  $t : X \rightarrow Y$  is a metacontext morphism and  $\rho$  is an  $V^*$ -indexed function from  $\Phi$  to  $\langle \text{MFrml}_L(\bar{s}(Y)) \mid \bar{s} \in V^* \rangle$ .

Every  $\langle t, \rho \rangle$  defines an *instantiation function*  $\{ - \}_{\langle t, \rho \rangle} : \text{SMFrml}_L(\bar{s}, X, \Phi) \rightarrow \text{MFrml}_L(\bar{s}(Y))$ . Due to the lack of space we are not able to present it here.<sup>7</sup> Using the above concept we are ready to define the notion of an *instantiation* for inference rules. Let  $r = \langle \langle \bar{s}, X, \Phi \rangle, \text{prem}(r), \text{conc}(r) \rangle$  be an  $L$ -inference-rule.

**Definition 13.** An instantiation of  $r$  in an  $L$ -metacontext  $Y$  via  $\langle t, \rho \rangle$  is a pair

$$\langle \{ \{ \varphi \}_{\langle t, \rho \rangle} \mid \varphi \in \text{prem}(r) \}, \{ \text{conc}(r) \}_{\langle t, \rho \rangle} \rangle.$$

In such a case we shall call the metaformula  $\{ \text{conc}(r) \}_{\langle t, \rho \rangle}$  an *immediate consequence* of  $\{ \{ \varphi \}_{\langle t, \rho \rangle} \mid \varphi \in \text{prem}(r) \}$  via  $r$  in  $Y$ .

For any (extended) presentation  $\mathcal{P}$  we can define a *sentence functor*  $\text{Sen}^{\mathcal{P}} : \text{SIG} \rightarrow \text{SET}$  such that

$$\text{Sen}^{\mathcal{P}}(\Sigma) \cong \bigcup_{X \in |\text{Ctx}_{\Sigma}^{\mathcal{P}}|} \{X\} \times \text{Frm}_{\Sigma}^{\mathcal{P}}(X)$$

for any signature  $\Sigma$  (with an obvious action on morphisms). In plain words,  $\Sigma$ -sentences are just “ $\Sigma$ -formulae with context”.

The inference rules of  $\mathcal{P}$  define an *entailment system*, i.e. a family  $\text{Ent}(\mathcal{P}) \cong \{ \vdash_{\Sigma}^{\mathcal{P}} \mid \Sigma \in |\text{SIG}| \}$  such that

- each  $\vdash_{\Sigma}^{\mathcal{P}} \subseteq \mathcal{P}(\text{Sen}_{\Sigma}^{\mathcal{P}}) \times \text{Sen}_{\Sigma}^{\mathcal{P}}$  and
- whenever  $\Gamma \vdash_{\Sigma_1}^{\mathcal{P}} \phi$  and  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  then  $\text{Sen}_{\sigma}^{\mathcal{P}}(\Gamma) \vdash_{\Sigma_2}^{\mathcal{P}} \text{Sen}_{\sigma}^{\mathcal{P}}(\phi)$ .

The entailment system  $\text{Ent}(\mathcal{P})$  is given as follows:  $\Gamma \vdash_{\Sigma}^{\mathcal{P}} \phi$  iff there exists a finite sequence  $\phi_1, \dots, \phi_n$  of  $\Sigma$ -sentences such that

- $\phi_n = \phi$

<sup>7</sup> The definition is quite obvious, with the only “delicate” point being the normalization requirement for the result.

- for all  $i \leq n$  either  $\phi_i \in \Gamma$  or  $\phi_i$  is an immediate consequence of some of the sentences from  $\{\phi_1, \dots, \phi_{i-1}\}$  via one of the rules from  $R_\Sigma$ <sup>8</sup> or  $\phi_i$  is a *substitution instance* of one of the sentences from  $\{\phi_1, \dots, \phi_{i-1}\}$ .<sup>9</sup>

It should be clear now why we have “omitted” the *substitution rule* from the Example 3. It is because the substitution rule is a built-in “metarule”, i.e. it is a part of our notion of *consequence* in the entailment system for  $\mathcal{P}$ .

For any signature  $\Sigma$  in SIG let us introduce the following *validity* satisfaction relation between  $\Sigma$ -models of  $\mathcal{P}$  and sentences from  $\text{Sen}^{\mathcal{P}}$ :

$$M \Vdash_\Sigma \langle X, \phi \rangle \quad \text{iff} \quad \text{for all } v \in \text{Val}_{\Sigma, M}^{\mathcal{I}(\mathcal{P})}(X) \quad M[v] \models_{\Sigma, X}^{\mathcal{I}(\mathcal{P})} \phi$$

Using the *semantic consequence* corresponding to  $\Vdash_\Sigma$  we can define the notion of *soundness* for rules. Let  $r$  be a  $\Sigma$ -rule in  $\mathcal{P}$ , i.e.  $r \in R_\Sigma$  where  $\mathcal{P}(\Sigma) = \langle \text{Lan}_\Sigma^{\mathcal{P}}, \text{Mod}_\Sigma^{\mathcal{P}}, \text{Int}_\Sigma, R_\Sigma \rangle$ .

**Definition 14.** *The rule  $r$  is sound iff for every  $\Sigma$ -context  $X$  and every  $\Xi \subseteq \text{Frm}_\Sigma^{\mathcal{P}}(X)$ ,  $\varphi \in \text{Frm}_\Sigma^{\mathcal{P}}(X)$ , if  $\varphi$  is an immediate consequence of  $\Xi$  via  $r$  in  $X$ , then  $\{\langle X, \xi \rangle \mid \xi \in \Xi\} \Vdash_\Sigma \langle X, \varphi \rangle$ .*

Using the validity satisfaction we can also easily define the *semantic soundness* and *semantic completeness* of the entailment system for  $\mathcal{P}$  given above.

**Definition 15.** *The entailment system  $\text{Ent}(\mathcal{P})$  is semantically sound if from  $\Gamma \vdash_\Sigma^{\mathcal{P}} \phi$  it follows that  $\Gamma \Vdash_\Sigma \phi$  for every signature  $\Sigma$ ,  $\Gamma \subseteq \text{Sen}_\Sigma^{\mathcal{P}}$ , and  $\phi \in \text{Sen}_\Sigma^{\mathcal{P}}$ . It is semantically complete if the implication in the other direction holds.*

The following proposition verifies that our notion of entailment for  $\mathcal{P}$  is intuitively “correct”.

**Proposition 2.** *The entailment system  $\text{Ent}(\mathcal{P})$  is semantically sound, if and only if, all its rules are sound.*

## 4.2 Putting extended presentations together

The category of *extended logical presentations* ELOGPRES is not complete (for the same reasons as LOGPRES is not complete). Therefore if we want to combine extended logical presentations, we will have to use a broader category EPRES for this purpose.<sup>10</sup>

<sup>8</sup> Strictly speaking, the metaformula being the second component of  $\phi$  has to be an immediate consequence of some of the metaformulae corresponding to the sentences  $\{\phi_1, \dots, \phi_{i-1}\}$ . This requires of course these premises and  $\phi$  to have a common metacontext.

<sup>9</sup> It means that there is an index  $k < i$  and a context morphism  $t : X \rightarrow Y$  in  $\text{Ctx}_\Sigma^{\mathcal{P}}$ , where  $\text{Ctx}_\Sigma^{\mathcal{P}} = \text{base}(\text{Frm}_\Sigma^{\mathcal{P}})$ , such that  $\text{Frm}_\Sigma^{\mathcal{P}}(t)(\varphi_k) = \varphi$  and  $\phi_k = \langle X, \varphi_k \rangle$  and  $\phi = \langle Y, \varphi \rangle$ .

<sup>10</sup> Limits in EPRES are just limits in PRES extended by the combined inference rules. Therefore all the observations about combining ordinary presentations remain true from the case of EPRES (cf. [9]).

The interesting thing is, how the inference rules behave in the combination process. Let us take the following example of a pullback square in  $\text{ePRES}$ :

$$\begin{array}{ccc}
 \mathcal{FOL}\mathcal{E}Q & \xrightarrow{\langle G', \nu' \rangle} & \mathcal{FOL} \\
 \downarrow \langle F', \mu' \rangle & & \downarrow \langle F, \mu \rangle \\
 \mathcal{E}Q & \xrightarrow{\langle G, \nu \rangle} & \mathcal{A}L\mathcal{G}
 \end{array}$$

As it turns out, the presentation  $\mathcal{FOL}\mathcal{E}Q$  is logical and the resulting entailment system is *semantically sound*. However, no matter how complete both  $\text{Ent}(\mathcal{E}Q)$  and  $\text{Ent}(\mathcal{FOL})$  are, their combination is *not complete*. For example let

$$\phi = \langle X, t_1 \equiv t_2 \Rightarrow (P(t_1) \Rightarrow P(t_2)) \rangle$$

where  $P$  is a unary predicate symbol and  $t_i$ 's are terms. Then  $\emptyset \Vdash \phi$ , but obviously  $\phi$  cannot be deduced from the empty set in  $\text{Ent}(\mathcal{FOL}\mathcal{E}Q)$ . The problem is that the nature of the interaction between the equality and other formulae is not described by any of the rules from the component presentations. So we cannot hope for a general ‘‘completeness preservation’’ result here. Fortunately, as one can show, the situation with soundness is better.

**Proposition 3.** *If the limit in  $\text{ePRES}$  of a diagram  $D$ , consisting of extended logical presentations only, is logical, then it preserves the soundness of the component inference systems.*

## 5 Concluding remarks

We have extended the notion of *context presentation* from [8, 9] by augmenting it by *inference rules*. The resulting framework seems to cover many interesting examples, although—due to the lack of space—only two simple ones were briefly sketched here.

For any *extended presentation*  $\mathcal{P}$  we have defined an entailment system  $\text{Ent}(\mathcal{P})$  generated by the rules of  $\mathcal{P}$  and the corresponding notions of *soundness* and *completeness* wrt. the context institution  $\mathcal{I}(\mathcal{P})$ .

Thanks to the completeness of the category  $\text{ePRES}$  of *extended presentations* we can use the standard categorical ‘‘machinery’’ as a tool for building logical systems with inference rules in a compositional way. Of course the process is rather far from being ‘‘automatic’’ (except for the simple cases). Assuming that both the components and the result are *logical* we can assure *soundness preservation*

The framework as presented, although quite powerful already, does not allow rules with *schematic substitution* such as the induction principle for example. We believe that adding them is possible and moreover, since the notion of *substitution* is a first-order citizen in context presentations, the extended framework should enjoy analogous meta-properties as the ones discussed above.

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